Theory of quantum channels for quantum networks: from bosonic modes to single photon

Quntao Zhuang*
Department of Electrical and Computer Engineering,
University of Arizona, Tucson, Arizona 85721, USA
James C. Wyant College of Optical Sciences, University of Arizona, Tucson, Arizona 85721, USA and
Ming Hsieh Department of Electrical and Computer Engineering,
University of Southern California, Los Angeles, California 90089, USA

Anthony J. Brady†
Department of Electrical and Computer Engineering,
University of Arizona, Tucson, Arizona 85721, USA

In this course, we will review the quantum channel description of light propagation in fiber or open space links. We will include both the bosonic mode case and the single-photon case. These basic channel models enable more detailed analyses for quantum information processing with photons.

Planned time: 3 hours + optional half hour discussion

CONTENTS

* zhuangquntao@gmail.com
† ajbrad4123@gmail.com

I. QUANTUM CHANNELS: GENERAL DESCRIPTION

Here, we discuss generic features of quantum evolution in terms of quantum channels and their properties. In later sections, we provide physical examples of quantum channels relevant for quantum information processing with photons.

A. Choi-Kraus representation

Consider a (finite or infinite dimensional) Hilbert space $\mathscr{H}$ and density matrices $\rho, \sigma \in \mathscr{H}$. We generally define a *quantum channel* $\mathcal{N} : \rho \to \sigma$ as a linear, completely positive, and trace preserving (CPTP) map. The map must be linear since quantum theory is linear, while the CPTP condition must hold because a quantum channel must 'map quantum states to quantum states'—i.e., $\sigma = \mathcal{N}(\rho) \geq 0$ if $\rho \geq 0$ and $\mathrm{Tr}\{\mathcal{N}(\rho)\} = \mathrm{Tr}\{\rho\}$. From linearity and the CPTP condition, one can prove the following useful theorem for any quantum channel $\mathcal{N}$ (see Section 4.4 of the publicly available book [1] for an explicit proof):

Theorem 1 (Choi-Kraus) *A map $\mathcal{N}$ is a CPTP map iff it admits a Choi-Kraus (or operator sum) representation as*

$$\mathcal{N}(\rho) = \sum_k \hat{L}_k \rho \hat{L}_k^\dagger, \tag{1}$$

where $\sum_k \hat{L}_k^\dagger \hat{L}_k = \hat{I}$.

The operators $\{\hat{L}_k\}$ are known as the *Kraus operators* for the quantum channel $\mathcal{N}$. Note that a quantum channel generalizes evolution of a quantum state beyond typical unitary evolution. Indeed, unitary evolution is a special case of the quantum channel description above, in which case a unitary quantum channel $\mathcal{U}(\rho) = \hat{U}\rho\hat{U}^\dagger$ has only one Kraus operator $\hat{U}$ which satisfies $\hat{U}^\dagger\hat{U} = \hat{I}$.

Exercise 1 (Kraus ops: Erasure) *Consider a two-level quantum system (a qubit) described by the quantum state $\Psi \in \mathscr{H}$, where $\mathscr{H}$ is the qubit Hilbert space, and consider the "erasure state" $|\varepsilon\rangle$ which lies outside of $\mathscr{H}$ (i.e., $\langle\varepsilon|\Psi|\varepsilon\rangle = 0 \,\forall\, \Psi \in \mathscr{H}$). An erasure channel $\mathcal{L}_\varepsilon$ acts on the qubit as,*

$$\mathcal{L}_\varepsilon(\Psi) = (1-\varepsilon)\Psi + \varepsilon\,|\varepsilon\rangle\langle\varepsilon|\,,$$

where $0 \le \varepsilon \le 1$ is the erasure probability. Given an orthonormal qubit basis $\{|0\rangle, |1\rangle\}$, write the Kraus operators for the erasure channel in terms of the qubit basis and the erasure state.

Exercise 2 (Kraus ops: Depolarizing) *Given a qubit described by the state Ψ, a depolarizing channel Δ_p has the following action,*

$$\Delta_p(\Psi) = (1-p)\Psi + p\hat{I}/2,$$

where $\hat{I}/2$ is the maximally mixed state and $0 \le p \le 4/3$. What are the Kraus operators for the depolarizing channel? [Hint: first show that, for any qubit Ψ,

$$\frac{1}{4}\left(\Psi + \hat{X}\Psi\hat{X} + \hat{Y}\Psi\hat{Y} + \hat{Z}\Psi\hat{Z}\right) = \hat{I}/2,$$

where $\hat{X}$, $\hat{Y}$, $\hat{Z}$ are the Pauli matrices. This is known as a "Pauli twirl".]

Exercise 3 (Concatenated erasures) *Consider two erasure channels $\mathcal{L}_{\varepsilon_1}$ and $\mathcal{L}_{\varepsilon_2}$ (see Exercise 1 for action of erasure channel on a qubit). Show that the concatenation of the two erasure channels is another erasure channel, $\mathcal{L}_{\varepsilon_{12}} = \mathcal{L}_{\varepsilon_2} \circ \mathcal{L}_{\varepsilon_1}$. Determine the erasure probability ε_{12}.*

B. Church of the Larger Hilbert Space

On the one hand, the postulates of quantum mechanics are written in terms of pure states $|\Psi\rangle$, which are complex vectors in a Hilbert space $\mathscr{H}$, and unitary operators which map pure states to pure states, i.e. $\hat{U} : |\Psi\rangle \to |\Psi'\rangle$. On the other hand, in the previous section, we discussed things in terms of a density matrix ρ, which is a convex (i.e., probabilistic) combination of pure states, and a quantum channel $\mathcal{N}$, which is generally non-unitary. We can reconcile these descriptions by 'going to the Church of the Larger Hilbert Space', in which case we describe a mixed quantum state by a pure quantum state in a larger Hilbert space and a quantum channel by a unitary evolution on a larger, joint system (formally known as Stinespring's dilation theorem). Details regarding such a reconciliation in terms of Kraus operators are discussed below.

1. Purification

Given a mixed quantum state $\rho_S \in \mathscr{H}_S$, where S denotes the 'system', we can introduce an environment Hilbert space $\mathscr{H}_E$ and formally construct a *purification* $|\Psi\rangle_{SE} \in \mathscr{H}_S \otimes \mathscr{H}_E$ of ρ_S such that,

$$\rho_S = \mathrm{Tr}_E\left\{\,|\Psi\rangle\langle\Psi|_{SE}\,\right\}, \tag{2}$$

where $\mathrm{Tr}_E\{\cdot\}$ denotes a partial trace over the environment degrees of freedom.

Explicitly, given the singular value decomposition, $\rho_S = \sum_i \lambda_i\,|i\rangle\langle i|_S$—where $\lambda_i \in \mathbb{R}$, $\sum_i \lambda_i = 1$, and $\langle i|j\rangle = \delta_{ij}$—a purification can be given as,

$$|\Psi\rangle_{SE} = \sum_i \sqrt{\lambda_i}\,|i,i\rangle_{SE}\,, \tag{3}$$

such that Eq. (2) is satisfied by construction. In this extended setting, the coefficients $\{\sqrt{\lambda_i}\}$ are known as *Schmidt coefficients*, and the system S and environment E are said to be entangled iff there is more than one Schmidt coefficient. Furthermore, the purification above is not unique, since unitary operations acting on the environment Hilbert space lead to the same system state ρ_S, i.e. $\hat{I}_S \otimes \hat{U}_E\,|\Psi\rangle_{SE} \implies \rho_S$.

2. Unitary and isometric extensions

As mentioned previously, we can reconcile quantum evolution via a quantum channel with unitary evolution by introducing a unitary interaction between a system S and an environment E, which is known as a *unitary extension* of the quantum channel.

Let $\mathcal{N}$ be a quantum channel with Kraus operators $\{\hat{L}_k\}$. Now, consider a system described by the density matrix ρ_S and an environment initially in the pure state $\sigma_E = |e_0\rangle\langle e_0|_E$, where $\langle e_0|e_0\rangle = 1$. The initial state of the environment can always be taken as pure since, if it were a mixed state, we could otherwise purify the state by introducing a larger environment Hilbert space. Then, given a set of basis vectors $\{|e_k\rangle_E\}$ and a joint unitary interaction $\hat{U}_{SE}$, we can explicitly construct the Kraus operators $\{\hat{L}_k\}$ via,

$$\begin{aligned}\mathcal{N}(\rho_S) &= \mathrm{Tr}_E\left\{\hat{U}_{SE}(\rho_S \otimes \sigma_E)\hat{U}_{SE}^\dagger\right\}\\ &= \sum_k \langle e_k|_E \left(\hat{U}_{SE}(\rho_S \otimes |e_0\rangle\langle e_0|_E)\hat{U}_{SE}^\dagger\right)|e_k\rangle_E\\ &= \sum_k \left(\langle e_k|\,\hat{U}_{SE}\,|e_0\rangle\right)\rho_S\left(\langle e_0|\,\hat{U}_{SE}^\dagger\,|e_k\rangle\right)\\ &= \sum_k \hat{L}_k\rho_S\hat{L}_k^\dagger,\end{aligned} \tag{4}$$

where $\hat{L}_k = \langle e_k|\,\hat{U}_{SE}\,|e_0\rangle$ are the Kraus operators that act on the system Hilbert space $\mathscr{H}_S$. By the completeness relation $\sum_k |e_k\rangle\langle e_k|_E = \hat{I}_E$ and the normality condition

$\langle e_0|e_0\rangle = 1$, the Kraus operators automatically satisfy $\sum_k \hat{L}_k^\dagger \hat{L}_k = \hat{I}_S$. Observe that the Kraus decomposition for the channel is not unique, as an arbitrary unitary transformation on the environment basis vectors lead to a different set of Kraus operators but describe the same channel.

Another useful representation is an *isometric extension* of a quantum channel. An isometry is a map $\hat{V}_{S\to SE} : \mathscr{H}_S \to \mathscr{H}_S \otimes \mathscr{H}_E$ such that $\hat{V}^\dagger_{S\to SE}\hat{V}_{S\to SE} = \hat{I}_S$ and $\hat{V}_{S\to SE}\hat{V}^\dagger_{S\to SE} = \hat{\Pi}_{SE}$, where $\hat{\Pi}_{SE}$ is a projection onto the joint Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_E$ (i.e., $\hat{\Pi}_{SE}$ satisfies $\hat{\Pi}^2_{SE} = \hat{\Pi}_{SE}$). Given a quantum channel $\mathcal{N}$ with Kraus operators $\{\hat{L}_k\}$ and environment basis vectors $\{|e_k\rangle_E\}$, one can construct an isometry

$$\hat{V}_{S\to SE} = \sum_k \hat{L}_k \otimes |e_k\rangle\,, \tag{5}$$

such that

$$\begin{aligned}\mathcal{N}(\rho_S) &= \mathrm{Tr}_E\left\{\hat{V}_{S\to SE}\rho_S\hat{V}^\dagger_{S\to SE}\right\}\\ &= \sum_{j,k}\hat{L}_k\rho_S\hat{L}_j^\dagger \underbrace{\langle e_k|e_j\rangle}_{=\delta_{kj}}\\ &= \sum_k \hat{L}_k\rho_S\hat{L}_k^\dagger.\end{aligned} \tag{6}$$

From the orthonormality of the environment basis vectors and the property $\sum_k \hat{L}_k^\dagger\hat{L}_k = \hat{I}$, one can indeed show that the isometric extension given in Eq. (5) satisfies $\hat{V}^\dagger_{S\to SE}\hat{V}_{S\to SE} = \hat{I}_S$ and, thus, that $\hat{V}_{S\to SE}\hat{V}^\dagger_{S\to SE}$ is a projector onto $\mathscr{H}_S \otimes \mathscr{H}_E$. Similar to the unitary extension discussed previously, all isometries are equivalent up to unitary transformations on the environment Hilbert space. The usefulness of the isometry is that the initial state of the environment need not be specified.

Exercise 4 (Isometric erasure) *Consider the erasure channel $\mathcal{L}_\varepsilon$. Provide an isometric extension of the erasure channel in terms of the qubit basis $\{|0\rangle, |1\rangle\}$, the erasure state $|\varepsilon\rangle$, and the erasure probability ε. [Hint: use the Kraus operators from Exercise 1 and Eq. (5).]*

II. GAUSSIAN BOSONIC CHANNELS

The discussion about quantum channels in the previous section was quite general, with not much reference to a particular quantum system. From hereon though, we shall focus on a one-dimensional quantum harmonic oscillator (e.g., photons with one degree of freedom). We describe the single bosonic mode by its annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, such that $[\hat{a}, \hat{a}^\dagger] = 1$. Necessary tidbits on the quantum harmonic oscillator can be found in Appendix A. Below, we define some typical Gaussian bosonic noise channels and then consider their properties in following subsections.

Definition 1 (Displacement channel) *Consider the displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, where $\alpha \in \mathbb{C}$. It acts on the annihilation operator $\hat{a}$ as*

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha. \tag{7}$$

We define the unitary displacement channel, $\mathcal{D}_\alpha$, as a unitary conjugation by $\hat{D}(\alpha)$. In other words, for a quantum state Ψ, $\mathcal{D}_\alpha(\Psi) = \hat{D}(\alpha)\Psi\hat{D}^\dagger(\alpha)$, which leads to Eq. (7) in the Heisenberg picture.

Definition 2 (Thermal loss channel) *Define a thermal loss channel $\mathcal{L}_{\eta,N_B}$ by the input-output relation for the annihilation operators,*

$$\hat{a}' = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e}, \tag{8}$$

where $\langle \hat{e}^\dagger\hat{e}\rangle = N_B$ is the mean number of bath quanta and $0 \le \eta \le 1$ is the transmittance. The output mean photon number is $\langle \hat{a}'^\dagger\hat{a}'\rangle = \eta\langle\hat{a}^\dagger\hat{a}\rangle + (1-\eta)N_B$. Observe that, for $N_B = 0$ (often called a pure-loss channel), $\langle \hat{a}'^\dagger\hat{a}'\rangle = \eta\langle\hat{a}^\dagger\hat{a}\rangle < \langle\hat{a}^\dagger\hat{a}\rangle$; i.e., photons have been lost.

Definition 3 (Thermal amplifier channel) *Define a thermal amplifier channel $\mathcal{A}_{G,N_B}$*

$$\hat{a}' = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}^\dagger, \tag{9}$$

where $\langle\hat{e}^\dagger\hat{e}\rangle = N_B$. The output mean photon number $\langle\hat{a}'^\dagger\hat{a}'\rangle = G\langle\hat{a}^\dagger\hat{a}\rangle + (G-1)(N_B+1)$. Observe that $\langle\hat{a}'^\dagger\hat{a}'\rangle \ge \langle\hat{a}^\dagger\hat{a}\rangle$; thus the intensity has been amplified. For $N_B = 0$, the channel is often referred to as a quantum limited amplifier.

Definition 4 (AGN channel) *Define an additive Gaussian noise (AGN) channel formally via*

$$\mathcal{N}_{N_B} = \lim_{\eta\to 1}\mathcal{L}_{\eta, N_B/(1-\eta)}. \tag{10}$$

The output mean photon number is $\langle\hat{a}'^\dagger\hat{a}'\rangle = \langle\hat{a}^\dagger\hat{a}\rangle + N_B$. Equivalently, one can define an AGN channel via Gaussian random displacements $\mathcal{D}_\xi$, where $\xi \sim \mathcal{N}_\mathcal{C}(0, N_B)$ and $\mathcal{N}_\mathcal{C}(0, N_B)$ is a (complex) normal distribution with variance N_B; in other words,

$$\mathcal{N}_{N_B}(\Psi) = \frac{1}{N_B\pi}\int_{\xi\in\mathbb{C}} \mathrm{d}^2\xi\, \mathrm{e}^{-\frac{|\xi|^2}{N_B}}\mathcal{D}_\xi(\Psi)\,. \tag{11}$$

Exercise 5 (AGN quanta) *Using Eq. (11), for any state Ψ, explicitly show that,*

$$\mathrm{Tr}\{\hat{n}\,\mathcal{N}_{N_B}(\Psi)\} = \mathrm{Tr}\{\Psi\} + N_B, \tag{12}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$.

A. Concatenated thermal loss

We consider the resulting channel from applying two thermal loss channels in succession. First apply $\mathcal{L}_{\eta_1,N_1}$ such that,

$$\hat{a}' = \sqrt{\eta_1}\hat{a} + \sqrt{1-\eta_1}\hat{e}_1. \tag{13}$$

Then apply $\mathcal{L}_{\eta_2,N_2}$ such that

$$\begin{aligned}\hat{a}'' &= \sqrt{\eta_2}\hat{a}' + \sqrt{1-\eta_2}\hat{e}_2 \\ &= \sqrt{\eta_2}(\sqrt{\eta_1}\hat{a} + \sqrt{1-\eta_1}\hat{e}_1) + \sqrt{1-\eta_2}\hat{e}_2 \\ &= \sqrt{\eta_1\eta_2}\hat{a} + \sqrt{1-\eta_1\eta_2}\hat{e}_3, \end{aligned} \tag{14}$$

where

$$\hat{e}_3 = \frac{\sqrt{\eta_2(1-\eta_1)}\hat{e}_1 + \sqrt{1-\eta_2}\hat{e}_2}{\sqrt{1-\eta_1\eta_2}} \tag{15}$$

and $\left[\hat{e}_3, \hat{e}_3^\dagger\right] = 1$. From here, it follows that

$$N_3 \equiv \left\langle \hat{e}_3^\dagger \hat{e}_3 \right\rangle = \frac{\eta_2(1-\eta_1)N_1 + (1-\eta_2)N_2}{1-\eta_1\eta_2} \tag{16}$$

We thus conclude that,

$$\mathcal{L}_{\eta_2,N_2} \circ \mathcal{L}_{\eta_1,N_1} = \mathcal{L}_{\eta_1\eta_2,N_3}. \tag{17}$$

B. Loss then amplifier

Some interesting things begin to happen once we start mixing concatenating different channels. Here we consider the channel resulting from amplification after loss, which is a standard setup for combat attenuation in, e.g., optical fibers.

First apply the thermal loss channel $\mathcal{L}_{\eta,N_1}$ such that

$$\hat{a}' = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e}_1. \tag{18}$$

Subsequently applying the thermal amplifier channel $\mathcal{A}_{G,N_2}$, we have

$$\begin{aligned}\hat{a}'' &= \sqrt{G}\hat{a}' + \sqrt{G-1}\hat{e_2}^\dagger \\ &= \sqrt{G}(\sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e}_1) + \sqrt{G-1}\hat{e_2}^\dagger \\ &= \sqrt{G\eta}\hat{a} + \left(\sqrt{G}\sqrt{1-\eta}\hat{e}_1 + \sqrt{G-1}\hat{e_2}^\dagger\right). \end{aligned} \tag{19}$$

Interestingly, the form of the resulting channel depends on the parameter values of G and η.

1. $G\eta < 1$: *Thermal-loss*

For $G\eta < 1$, the resulting channel is a thermal loss channel. In particular, we rewrite expression (19) as

$$\hat{a}'' = \sqrt{G\eta}\hat{a} + \sqrt{1-G\eta}\hat{e_3}. \tag{20}$$

with

$$\hat{e_3} = \left(\sqrt{\frac{G(1-\eta)}{1-G\eta}}\hat{e}_1 + \sqrt{\frac{G-1}{1-G\eta}}\hat{e_2}^\dagger\right). \tag{21}$$

One can check that $[\hat{e_3}, \hat{e_3}^\dagger] = 1$ and

$$N_3 \equiv \left\langle \hat{e_3}^\dagger \hat{e_3} \right\rangle = \frac{G(1-\eta)}{1-G\eta}N_1 + \frac{G-1}{1-G\eta}(N_2+1). \tag{22}$$

For $G\eta < 1$, the channel resulting from the composition (thermal loss then thermal amplifier) is thus a thermal loss channel, i.e.

$$\mathcal{A}_{G,N_2} \circ \mathcal{L}_{\eta,N_1} = \mathcal{L}_{G\eta,N_3} \quad (G\eta < 1). \tag{23}$$

2. $G\eta = 1$: *AGN*

In this case, we can take the limit of the composition above, i.e.

$$N_B = \lim_{\eta G \to 1}(1-G\eta)N_3 = (G-1)(N_1+N_2+1). \tag{24}$$

Therefore,

$$\mathcal{A}_{G,N_2} \circ \mathcal{L}_{\eta,N_1} = \mathcal{N}_{N_B} \quad (G\eta = 1). \tag{25}$$

3. $G\eta > 1$: *Thermal amplifier*

For $G\eta > 1$, the resulting channel is a thermal amplifier channel. In particular, we rewrite expression (19) as

$$\hat{a}'' = \sqrt{G\eta}\hat{a} + \sqrt{G\eta-1}\hat{e}_4^\dagger, \tag{26}$$

with

$$\hat{e}_4 = \left(\sqrt{\frac{G(1-\eta)}{G\eta-1}}\hat{e}_1^\dagger + \sqrt{\frac{G-1}{G\eta-1}}\hat{e_2}\right). \tag{27}$$

One can check that $[\hat{e}_4, \hat{e}_4^\dagger] = 1$ and

$$N_4 \equiv \left\langle \hat{e}_4^\dagger \hat{e}_4 \right\rangle = \frac{G(1-\eta)}{G\eta-1}(N_1+1) + \frac{G-1}{G\eta-1}N_2. \tag{28}$$

For $G\eta > 1$, the channel resulting from the composition (thermal loss then thermal amplifier) is thus a thermal amplifier channel, i.e.

$$\mathcal{A}_{G,N_2} \circ \mathcal{L}_{\eta,N_1} = \mathcal{A}_{G\eta,N_4}, \text{for } G\eta > 1. \tag{29}$$

C. Amplifier then loss

Derivations and results are quite similar for this case. However, one interesting result is that, for $G\eta = 1$, amplification prior to loss introduces less noise than amplification after loss.

Exercise 6 (Amplifier then loss is less noisy) *Consider a thermal loss channel $\mathcal{L}_{\eta,N_2}$ and a thermal amplifier channel $\mathcal{A}_{G,N_1}$. For $G\eta = 1$, show that amplification prior to loss introduces less noise than* *amplification after loss. [Hint: One must first show that, for $G\eta = 1$, amplification before loss results in an AGN channel similar to Eq. (25), i.e.*

$$\mathcal{L}_{\eta,N_2} \circ \mathcal{A}_{G,N_1} = \mathcal{N}_{N_{B_1}} \quad (G\eta = 1). \tag{30}$$

Find N_{B_1}. Then, given $\mathcal{A}_{G,N_1} \circ \mathcal{L}_{\eta,N_2} = \mathcal{N}_{N_{B_2}}$, show $N_{B_1} < N_{B_2}$, where N_{B_2} can be taken from Eq. (24).]

D. Unitary extensions of Gaussian channels

It turns out that there exists very simple classification schemes for every possible single-mode Gaussian bosonic channel [2, 3] with simple corresponding unitary extensions, however we will not go into such technical details in this course. Instead, we provide heuristic explanations for the thermal loss channel and thermal amplifier channel.

The thermal loss channel $\mathcal{L}_{\eta,N_B}$ and thermal amplifier channel $\mathcal{A}_{G,N_B}$ are non-unitary channels. However, we can provide an unitary extension of such channels by introducing an ancillary mode $\hat{e}$ that occupies a thermal state and interacts with the system mode by a two-mode, Gaussian unitary operation $\hat{U}_{SE}$.

1. Loss via beamsplitter

The unitary extension of a loss channel $\mathcal{L}_{\eta,N_B}$ is a two-mode beamsplitter with transmittance η. The beamsplitter unitary is,

$$\hat{U}_{SE}(\theta) = \exp\left[i\theta(\hat{a}^\dagger\hat{e} - \hat{a}\hat{e}^\dagger\right], \tag{31}$$

where the beamsplitter angle θ is related to the transmittance η via $\eta = \cos^2\theta$. The system mode $\hat{a}$ and environment mode $\hat{e}$ evolve under the beamsplitter unitary of Eq. (31) via

$$\hat{a}' \equiv \hat{U}^\dagger_{SE}(\theta)\hat{a}\hat{U}_{SE}(\theta) = \cos\theta\,\hat{a} + \sin\theta\,\hat{e}, \tag{32}$$

$$\hat{e}' \equiv \hat{U}^\dagger_{SE}(\theta)\hat{e}\hat{U}_{SE}(\theta) = \cos\theta\,\hat{e} - \sin\theta\,\hat{a}. \tag{33}$$

Exercise 7 (Beamsplitter: Output quanta) *Assume that the environment mode $\hat{e}$ is a (Gaussian) thermal state with zero mean, $\langle\hat{e}\rangle = 0$, and $N_B \geq 0$ number of quanta, $\langle\hat{e}^\dagger\hat{e}\rangle = N_B$. Using Eq. (32), show that $\langle\hat{a}'^\dagger\hat{a}'\rangle = \eta\langle\hat{a}^\dagger\hat{a}\rangle + (1-\eta)N_B$, where $\eta = \cos^2\theta$.*

The pure-loss channel $\mathcal{L}_{\eta,0}$ corresponds to the special case that the environment mode $\hat{e}$ occupies a vacuum state with zero quanta rather than a thermal state of N_B quanta. For optical frequencies, the pure-loss channel is often a good approximation to use in place a general thermal loss channel.

2. Amplification via two-mode squeezing

The unitary extension of a thermal amplifier channel $\mathcal{A}_{G,N_B}$ is a two-mode squeezer. The two-mode squeezing unitary is,

$$\hat{U}_{SE}(r) = \exp\left[r(\hat{a}\hat{e} - \hat{a}^\dagger\hat{e}^\dagger)\right], \tag{34}$$

where the squeezing strength r is related to the gain G of the amplifier channel via $G = \cosh^2 r$. The system mode $\hat{a}$ and environment mode $\hat{e}$ evolve under the two-mode squeezing unitary of Eq. (34) via

$$\hat{a}' \equiv \hat{U}^\dagger_{SE}(r)\hat{a}\hat{U}_{SE}(r) = \cosh r\,\hat{a} + \sinh r\,\hat{e}^\dagger, \tag{35}$$

$$\hat{e}' \equiv \hat{U}^\dagger_{SE}(r)\hat{e}\hat{U}_{SE}(r) = \cosh r\,\hat{e} + \sinh r\,\hat{a}^\dagger, \tag{36}$$

where $\cosh^2 r - \sinh^2 r = 1$. Observe that the total number of quanta $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{e}^\dagger\hat{e}$ is not conserved under a two-mode squeezing operation. One can see this immediately by noting the presence of creation operators in the output relations for the annihilation operators above. Similarly, one can show that the output quanta $\hat{N}'$ does not equal the input quanta $\hat{N}$, i.e. $\hat{N}' \neq \hat{N}$.

Exercise 8 (Squeezing: Output quanta) *Assume that the environment mode $\hat{e}$ is a (Gaussian) thermal state with zero mean, $\langle\hat{e}\rangle = 0$, and $N_B \geq 0$ number of quanta, $\langle\hat{e}^\dagger\hat{e}\rangle = N_B$. Using Eq. (35), show that $\langle\hat{a}'^\dagger\hat{a}'\rangle = G\langle\hat{a}^\dagger\hat{a}\rangle + (G-1)(N_B+1)$, where $G = \cosh^2 r$. Thus, even for vacuum inputs $\langle\hat{a}^\dagger\hat{a}\rangle = \langle\hat{e}^\dagger\hat{e}\rangle = 0$, there are $G-1$ quanta in the output.*

A thermal amplifier channel with zero background quanta $\mathcal{A}_{G,0}$ is often called a quantum-limited amplifier, which corresponds to the special case that the environment mode $\hat{e}$ occupies a vacuum state. Quantum-limited amplifiers are desirable devices that can be used to (noisily) amplify both quadratures of very weak signals.

III. SINGLE-PHOTON ENCODINGS

Here, we discuss single-photon encodings via "dual rail" qubits and their relevance in quantum information processing. For further details, refer to, e.g., Chapter 5 of Ref. [4] and Refs. [5, 6].

A. Dual-rail qubits

Photons have many degrees of freedom (dofs)—polarization, angular momentum, spatial, temporal, etc.—with varying properties and levels of control. Some degrees of freedom are finite dimensional (e.g., a photon only has two modes of polarization), while others are infinite dimensional and either continuous (like spatial or temporal) or discrete (like angular momentum).

We will encode into only one dof, which we label with the set of mode operators $\{\hat{a}_k\}$. The index k refers to the mode of the dof, e.g. $k \in \{H, V\}$ for horizontally (H) or vertically (V) polarized light. Though k can generally run over an infinite set (such as angular momentum), for dual-rail photonic qubits, we restrict to a two mode subspace, generically labelled $k \in \{1, 2\}$.

We define a logical qubit as a state in the two-mode, single photon subspace with logical states given by

$$|0\rangle \equiv \hat{a}_1^\dagger |\text{vac}\rangle\,, \quad |1\rangle \equiv \hat{a}_2^\dagger |\text{vac}\rangle\,, \tag{37}$$

such that $\Psi \in \text{span}\{|0\rangle, |1\rangle\}$ for a general dual-rail qubit Ψ. [Note that, technically, $|\text{vac}\rangle = |\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$ and, e.g., $\hat{a}_1^\dagger |\text{vac}\rangle = \hat{a}_1^\dagger \otimes \hat{I} |\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$, but we drop this extra baggage for brevity.]

All single-qubit operations can be implemented with two-mode *passive operations* consisting of (unitary) beam splitters and phase-shifters, $\hat{U}_{\text{BS}}$ and $\hat{U}_\phi$, described by Hamiltonians

$$\hat{H}_{\text{BS}} = i\theta \mathrm{e}^{i\varphi} \hat{a}_1^\dagger \hat{a}_2 + \text{h.c.}, \tag{38}$$

$$\hat{H}_\phi = \sum_{k=1}^{2} \phi_k \hat{a}_k^\dagger \hat{a}_k. \tag{39}$$

These operations are called passive because they preserve the total photon number $\hat{N} \equiv \sum_k \hat{a}_k^\dagger \hat{a}_k$.

Exercise 9 (Passive Operations) *Show that any Hamiltonian of the form $\hat{H} = \sum_{j,k} H_{jk} \hat{a}_j^\dagger \hat{a}_k$ commutes with the total photon number operator $\hat{N}$.*

The action of a general beamsplitter is

$$\hat{a}_1' = \cos\theta\, \hat{a}_1 + \mathrm{e}^{i\varphi} \sin\theta\, \hat{a}_2, \tag{40}$$

$$\hat{a}_2' = \cos\theta\, \hat{a}_2 - \mathrm{e}^{-i\varphi} \sin\theta\, \hat{a}_1. \tag{41}$$

Writing a vector of operators $\hat{\boldsymbol{a}} = (\hat{a}_1, \hat{a}_2)$, we can compactly write the transformation as

$$\hat{\boldsymbol{a}}' = \underbrace{\begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix}}_{\equiv V_{\text{BS}}} \hat{\boldsymbol{a}} \tag{42}$$

where we have defined the 2×2 matrix V_{BS}. Note that $V_{\text{BS}}^\dagger V_{\text{BS}} = \hat{I}$ and $\det V_{\text{BS}} = 1$. Hence V_{BS} is a special unitary matrix, and, starting from the logical basis, one can implement any single-qubit rotation (up to a global phase) via V_{BS}. Indeed, one can show that

$$|0\rangle \xrightarrow{V_{\text{BS}}} \cos\theta\, |0\rangle + \mathrm{e}^{-i\varphi} \sin\theta\, |1\rangle\,, \tag{43}$$

$$|1\rangle \xrightarrow{V_{\text{BS}}} -\mathrm{e}^{i\varphi} \sin\theta\, |0\rangle + \cos\theta\, |1\rangle\,. \tag{44}$$

Exercise 10 (Pauli-X and -Y) *For what values of θ and φ can we implement Pauli-X and Pauli-Y matrices (up to a global phase), $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$?*

For the phase shifts generated by the Hamiltonian in Eq. (39), we have

$$V_\phi = \begin{pmatrix} \mathrm{e}^{i\phi_1} & 0 \\ 0 & \mathrm{e}^{i\phi_2} \end{pmatrix}. \tag{45}$$

Thus by choosing $\phi_1 = 0$ and $\phi_2 = \pi$, we have the Pauli-Z matrix $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Exercise 11 (Hadamard) *What combination of phase-shifters and beamsplitters produces the Hadamard matrix, $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$?*

B. Spatial, polarization, and time-bin encodings

The dof that we choose for encoding often depends on the context or application. Some key questions to consider when choosing a dof for encoding are:

- Is the dof easy to manipulate such that, e.g., arbitrary single-qubit operations are possible?
- Which dofs are more robust to practical noise sources?
- Can we scale up for large-scale quantum information processing with multiple qubits?

Answers to these questions vary with the context. Typical encodings used for quantum information processing tasks are (i) spatial, (ii) polarization, and (iii) time-bin.

(i) *Spatial*: Photon with fixed frequency ω, polarization etc., but may traverse two distinct paths labelled $k = 1, 2$. Interaction induced by overlapping the paths at, e.g., beamsplitters. Phase shifts induced by changing path lengths s.t. $\phi_k = \omega L_k / c$.

(ii) *Polarization*: Photon with fiexed frequency, spatial path etc., but may be in a superposition of polarization states. Horizontal H and vertical V polarization define logical states, $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$. Diagonal, circular polarizations given by superpositions of H and V. Birefringent materials used to implement single-photon operations.

(iii) *Time-bin*: Photon with fixed frequency, polarization, spatial path etc., but may occupy two distinct time-binned intervals $k = e, l$ (e for early, l for late). Fast optical switches and delays implement single-photon operations.

We can also swap between encodings. For instance, given two polarization modes H, V and two spatial modes $1, 2$, we can implement a polarizing beamsplitter (PBS) such that

$$\hat{a}_{H,1} \to \hat{a}_{H,1} \quad \text{and} \quad \hat{a}_{H,2} \to \hat{a}_{H,2}, \tag{46}$$

$$\hat{a}_{V,1} \to \hat{a}_{V,2} \quad \text{and} \quad \hat{a}_{V,2} \to \hat{a}_{V,1}. \tag{47}$$

In other words, the horizontal polarization gets transmitted through the beamsplitter while the vertical polarization gets reflected. Following the PBS, we can rotate, e.g., a V into an H, thus swapping a polarization qubit for a spatial qubit. A similar swap can be done between time-bin and polarization with fast optical switching [4].

IV. SINGLE-PHOTON EVOLUTION

Quantum information is often encoded into the degrees of freedom of a single photon, as discussed in the previous section. Moreover, most (quantum) communication links are over, e.g., noisy fibers or free-space links, which can be accurately described by thermal loss channels. Thus in this section, we will focus on the action of a thermal loss channel $\mathcal{L}_{\eta,N_B}$ on a single-photon state ρ_1. In physical scenarios, the background quanta N_B is equal to the population of the environment (originating from, e.g., the sun, the moon, or background lights for free-space links), whereas the loss probability $1-\eta$ of the channel is equal to the absorption probability of the medium. For instance, given a fiber of length L, $\eta = \mathrm{e}^{-\alpha L}$ where α is an attenuation coefficient (typically quoted in dB/km). The exponential attenuation is a consequence of the Beer-Lambert law for absorptive media.

Consider a thermal-loss channel $\mathcal{L}_{\eta,N_B}$ from Eq. (23). It admits the following decomposition

$$\mathcal{L}_{\eta,N_B} = \mathcal{A}_{G,0} \circ \mathcal{L}_{\tau,0}, \tag{48}$$

with

$$\tau G = \eta, \tag{49}$$

$$\frac{G-1}{1-G\tau} = N_B. \tag{50}$$

The parameters τ and G are related to η and N_B via

$$G = (1-\eta)N_B + 1, \tag{51}$$

$$\tau = \frac{\eta}{(1-\eta)N_B + 1}. \tag{52}$$

From above, a thermal loss channel can be decomposed into a pure loss channel and a quantum-limited amplifier. Thus the Kraus operators can be obtained as [7, 8]

$$\mathcal{L}_{\eta,N_B}(\rho) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \hat{B}_k \hat{A}_\ell \rho \hat{A}_\ell^\dagger \hat{B}_k^\dagger, \tag{53}$$

where

$$\hat{A}_\ell = \sqrt{\frac{(1-\tau)^\ell}{\ell!}}\tau^{\hat{a}^\dagger\hat{a}/2}\hat{a}^\ell \tag{54}$$

$$\hat{B}_k = \sqrt{\frac{1}{k!}\frac{1}{G}\left(\frac{G-1}{G}\right)^k}\hat{a}^{\dagger k} G^{-\hat{a}^\dagger\hat{a}/2}. \tag{55}$$

Note that $\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$.

Let's focus on a single photon input $\rho_1 = |1\rangle\langle 1|$. Terms $\hat{A}_\ell \rho_1 \hat{A}_\ell^\dagger$are only non-zero when $\ell = 0, 1$—i.e.,

$$\hat{A}_0 \left|1\right\rangle\left\langle 1\right| \hat{A}_0^\dagger = \tau \left|1\right\rangle\left\langle 1\right|. \tag{56}$$

and

$$\hat{A}_1 \left|1\right\rangle\left\langle 1\right| \hat{A}_1^\dagger = (1-\tau) \left|0\right\rangle\left\langle 0\right|. \tag{57}$$

This is very intuitive. With probability τ, the photon is transmitted. With probability $1-\tau$, the photon is lost.

Things get more complicated for the amplifier. We focus on the $k = 0, 1$ terms, with operators given as

$$\hat{B}_0 = \sqrt{\frac{1}{G}}G^{-\hat{a}^\dagger\hat{a}/2} \tag{58}$$

$$\hat{B}_1 = \sqrt{\frac{1}{G}\left(\frac{G-1}{G}\right)}\hat{a}^\dagger G^{-\hat{a}^\dagger\hat{a}/2}. \tag{59}$$

Composing these with the pure-loss channel from above,

$$\hat{B}_0\hat{A}_0 \left|1\right\rangle\left\langle 1\right| \hat{A}_0^\dagger \hat{B}_0 = \frac{\tau}{G^2}\left|1\right\rangle\left\langle 1\right|, \tag{60}$$

in which case the photon is unaffected by the channel. For the next term,

$$\hat{B}_1\hat{A}_0 \left|1\right\rangle\left\langle 1\right| \hat{A}_0^\dagger \hat{B}_1 = \frac{2(G-1)}{G^3}\tau\left|2\right\rangle\left\langle 2\right|, \tag{61}$$

in which case one noisy photon is added to the state. Going further,

$$\hat{B}_0\hat{A}_1 \left|1\right\rangle\left\langle 1\right| \hat{A}_1^\dagger \hat{B}_0 = \frac{(1-\tau)}{G}\left|0\right\rangle\left\langle 0\right|, \tag{62}$$

in which case the original photon is simply lost and we are left with vacuum. Finally,

$$\hat{B}_1\hat{A}_1 \left|1\right\rangle\left\langle 1\right| \hat{A}_1^\dagger \hat{B}_1 = \frac{G-1}{G^2}(1-\tau)\Theta_1, \tag{63}$$

where Θ_1 is a completely mixed (i.e., thermal) single-photon from the channel. [For a single photon with two degrees of freedom, $\Theta_1 = \hat{I}/2$, where $\hat{I}$ is the identity on the single-photon Hilbert space.] In this case, the initial photon is lost and then replaced with a noisy photon.

The overall channel, when acting on an arbitrary single-photon state ρ_1, can then be written as

$$\begin{aligned}\mathcal{L}_{\eta,N_B}(\rho_1) = &\frac{\tau}{G^2}\rho_1 + \frac{G-1}{G^2}(1-\tau)\Theta_1 \\ &+ \frac{(1-\tau)}{G}\left|\text{vac}\right\rangle\left\langle\text{vac}\right| \\ &+ \frac{(G-1)^2 + 2\tau(G-1)}{G^2}\rho_{\geq 2\text{ photons}},\end{aligned} \tag{64}$$

where $\rho_{\geq 2\text{ photons}}$ is a quantum state with more than two photons.

Exercise 12 (Pure loss and erasure) *Show that the pure loss channel $\mathcal{L}_{\eta,0}$, when acting on any single-photon state ρ_1, is equivalent to an erasure channel (see Exercise 1). Determine the erasure probability and the erasure state. [Hint: One can find a quick solution via equations above.]*

The probability of successfully transmitting the photon ρ_1 through the channel $\mathcal{L}_{\eta,N_B}$, is [using Eqs. (51) and (52)]

$$p_{\text{success}} = \frac{\tau}{G^2} = \frac{\eta}{\left[(1-\eta)N_B + 1\right]^3}. \tag{65}$$

The probability of getting a random single photon (i.e., depolarized single photon) is

$$p_{\text{depolarizing}} = \frac{G-1}{G^2}(1-\tau) = \frac{(1-\eta)^2 N_B(N_B+1)}{[(1-\eta)N_B+1]^3}. \tag{66}$$

The probability of getting nothing (i.e., the vacuum) is

$$p_{\text{vac}} = \frac{(1-\tau)}{G} = \frac{(1-\eta)(N_B+1)}{[(1-\eta)N_B+1]^2}. \tag{67}$$

Finally, the probability of getting two or more photons is

$$p_{\geq 2} = 1 - p_{\text{success}} - p_{\text{depolarizing}} - p_{\text{vac}}. \tag{68}$$

Exercise 13 (Low thermal noise) *Expand the output probabilities p_{success}, p_{vac}, and $p_{\text{depolarizing}}$ to first order in N_B and write out the resulting expressions. Show explicitly that $p_{\geq 2} = 2\eta(1-\eta)N_B + \mathcal{O}\big(N_B^2\big)$. Can you intuitively explain the result for $p_{\geq 2}$?*

Appendix A: Tidbits on the quantum harmonic oscillator

Consider the (normalized) position and momentum operators $\hat{q}$ and $\hat{p}$ which obey the canonical commutation relations

$$[\hat{q}, \hat{p}] = i\hat{I}. \tag{A1}$$

The annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$, are related to the canonical operators via

$$\hat{a} = \frac{1}{2}\left(\hat{q} + i\hat{p}\right), \tag{A2}$$

which obey $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$. Likewise, $\mathrm{Re}\{\hat{a}\} = \hat{q}$ and $\mathrm{Im}\{\hat{a}\} = \hat{p}$. We consider a Harmonic oscillator of frequency ω with Hamiltonian,

$$\begin{aligned} \hat{H} &= \frac{\hbar\omega}{2}\left(\hat{q}^2 + \hat{p}^2\right) && \text{(A3)} \\ &= \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega/2 && \text{(A4)} \\ &= \hbar\omega\hat{n} + \hbar\omega/2, && \text{(A5)} \end{aligned}$$

where the additive constant $\hbar\omega/2$ is the 'zero-point energy'. We ignore it for the most part. The operator $\hat{n} \equiv \hat{a}^\dagger\hat{a}$ is the number operator and tells us how many quanta occupy an oscillator state.

One can show that the following 'Fock states' are eigenstates of the number operator (and thus the oscillator Hamiltonian),

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\,|\text{vac}\rangle\,, \tag{A6}$$

where n is a positive integer and $|\text{vac}\rangle \equiv |n=0\rangle$ is the vacuum (i.e., lowest energy) state defined implicitly via $\hat{a}\,|\text{vac}\rangle = 0$. From the vacuum condition and the commutator for annihilation and creation operators, we have that

$$\hat{n}\,|n\rangle = n\,|n\rangle\,. \tag{A7}$$

The set $\{|n\rangle\}_{n=0}^{\infty}$ form a basis in the bosonic Hilbert space of a single mode (of mode frequency ω) $\mathscr{H}$, i.e.

$$\langle m|n\rangle = \delta_{mn} \tag{A8}$$

$$\text{and} \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}, \tag{A9}$$

where δ_{mn} is the Kronecker delta and $\hat{I}$ is the identity on $\mathscr{H}$. One can show that

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{A10}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{A11}$$

i.e. $\hat{a}$ annihilates one quanta and $\hat{a}^\dagger$ creates one quanta.

[1] M. M. Wilde, From classical to quantum shannon theory (2011), arXiv:1106.1445 [quant-ph].

[2] A. S. Holevo, One-mode quantum Gaussian channels: Structure and quantum capacity, Probl. Inf. Transm. **43**, 1 (2007).

[3] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[4] P. Kok and B. W. Lovett, *Introduction to Optical Quantum Information Processing* (Cambridge University Press, 2010).

[5] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, and G. J. Milburn, Linear optical quantum computing with photonic qubits, Rev. Mod. Phys. **79**, 135 (2007).

[6] C. Fabre and N. Treps, Modes and states in quantum optics, Rev. Mod. Phys. **92**, 035005 (2020).

[7] C. N. Gagatsos, B. A. Bash, S. Guha, and A. Datta, Bounding the quantum limits of precision for phase estimation with loss and thermal noise, Phys. Rev. A **96**, 062306 (2017).

[8] J. S. Ivan, K. K. Sabapathy, and R. Simon, Operator-sum representation for bosonic gaussian channels, Physical Review A **84**, 042311 (2011).