Center for
Quantum Networks
NSF Engineering Research Center

Theory of quantum channels for quantum networks

Instructor: Quntao Zhuang
– University of Southern California

Co-Instructor: Anthony J. Brady
— University of Arizona

★ This work is supported primarily by the Engineering Research Centers Program of the National Science Foundation. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect those of the National Science Foundation.

CQN Winter School on Quantum Networks

Funded by National Science Foundation Grant #1941583

Table of contents

INTRODUCTION AND MOTIVATION

The era of quantum engineering

LIGO

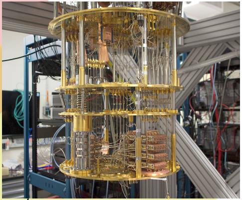
Quantum computer
IBM, >50 qubits,
Google supremacy,

Quantum sensing

Using non-classical source to enhance hypothesis testing and parameter estimation

. . . . . .

As sub-routine in communication and computing



Quantum communication

Quantum key distribution
Quantum teleportation
Quantum network
Entanglement-assisted communication
. . . . . .

Enabling non-classical resource at distance

Quantum satellite

Quantum computing

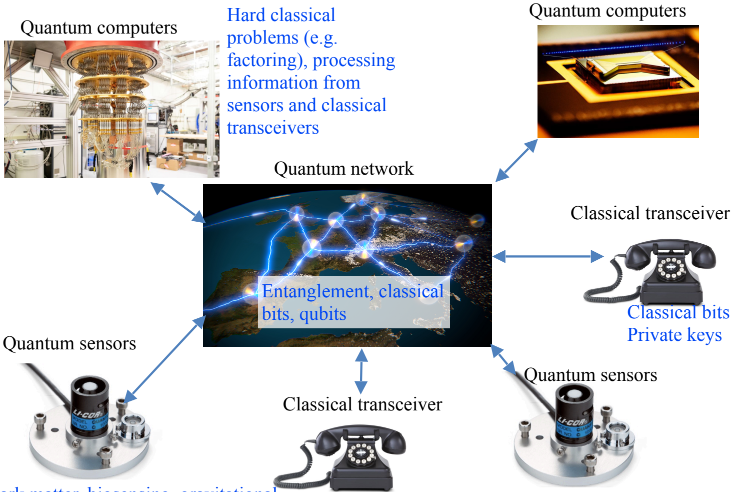
Quantum adiabatic algorithms
Quantum circuit model
Measurement-based quantum computing
NISQ quantum computation
Quantum machine learning

Enables receiver design,
Basic components

1

Center for Quantum Networks NSF-ERC

Quantum networking



Quantum computers

Hard classical problems (e.g. factoring), processing information from sensors and classical transceivers

Quantum computers

Quantum network

Entanglement, classical bits, qubits

Classical transceiver

Classical bits
Private keys

Quantum sensors

Quantum sensors

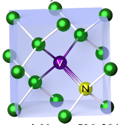
Classical transceiver

dark matter, biosensing, gravitational wave, radar, lidar, Spectroscopy

Many physical platforms and systems

(But we'll focus on photons...)

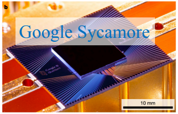
NV centers, color centers



Waldherr et al. Nature **506**, 204 (2014)

Long coherence time

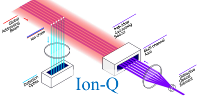
SC transmon qubits



Arute et al., Nature **574**, 505 (2019).

Large scale, strong interaction

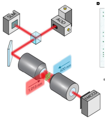
Trapped-ion



Wright et al., Nat. Commun. **10**, 5464 (2019)

High fidelity, high circuit volume

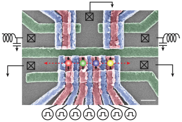
Rydberg atoms



Nature **595**, 227–232 (2021)

Large scale, high fidelity

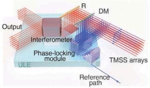
Electron spin in Q dots



Kandel et al., Nature **573** 553 (2019)

Long coherence time

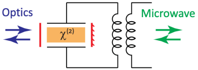
Bulk optics/nanophotonics



Zhong et al., Science **370** 1460 (2020)

Scalable, sensing/communication

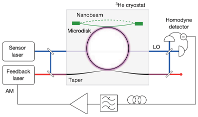
Transduction systems



Fan et al., Science Adv. **4** aar4994 (2018)

networking

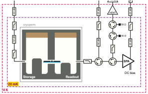
Optomechanical systems



Wilson et al., Nature **524** 325 (2015)

Force sensing

Microwave SC cavities



Campagne-Ibarcq et al.,
Nature **584**, 368 (2020)

Strong interaction, robustness

Center for Quantum Networks NSF-ERC

Poll 0: Photonic quantum information processing

What are photons good for?

(A) Quantum sensing
(B) Quantum computing
(C) Quantum communication
(D) All of the above

Answer 0: Photonic quantum information processing

What are photons good for?

(A) Quantum sensing

(B) Quantum computing

(C) Quantum communication

(D) All of the above

Photons are very versatile. Several quantum computing approaches exists that are based on linear optics, microwave cavity modes, etc. Quantum sensing platforms have been demonstrated with microwave cavities, quantum optical setups, etc. Finally, quantum information processing across long distances will require quantum optical interlinks.

QUANTUM CHANNELS: GENERAL DESCRIPTION

Quantum Physics principle:

our universe is elvoving under unitary dynamics.

→ if we include everything relevant, it's always unitary

→ if we include everything relevant, it's always pure.

example: thermalization

$$\text{system} \;-\!\!-\; \overbrace{\text{bath} + \text{anything else}}^{\text{environment}}$$

Mixed State : coming from partial trace

Universe: $S+E$ in a pure state $|\Psi\rangle_{SE}$

partial trace: $\rho_S = tr_E\left(|\Psi\rangle_{SE}\langle\Psi|\right)$

Schmidt decomposition: $|\Psi\rangle_{SE} = \sum_i \sqrt{\lambda_i}|i\rangle_S|i\rangle_E$
(singular value)

$$\rho_S = \sum_{|j\rangle_E} \langle j| \sum_{i_1} \sqrt{\lambda_{i_1}}|i_1\rangle_S|i_1\rangle_E \sum_{i_2}\sqrt{\lambda_{i_2}}\langle i_2|_S\langle i_2|_E |j\rangle$$

$$= \sum_j \underbrace{\lambda_j \; |j\rangle_S\langle j|}$$

eigenvalues of $\hat{\rho}_S$

Inverse the process: purification.

Universe: $S+E$ in a pure state $|\Psi\rangle_{SE}$

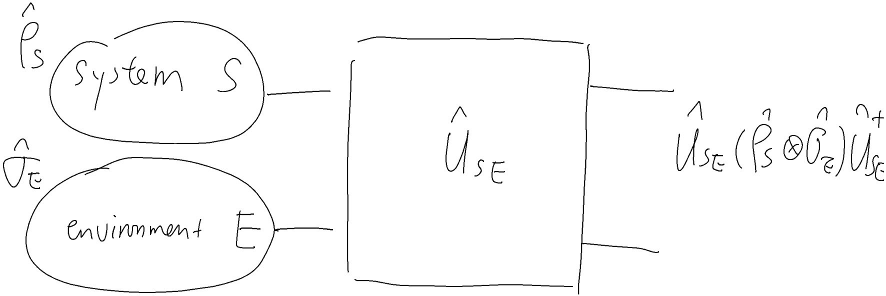
unitary degree of freedom: $|\Psi\rangle_{SE} \to \mathbb{1}_S \otimes \hat{U}_E |\Psi\rangle_{SE}$

joint state $|\Psi\rangle_{SE} = \sum_i \sqrt{\lambda_i}\, |i\rangle_S |i\rangle_E$

append environment. E, bases $\{|j\rangle_E\}$ → free to choose $\{U_E |j\rangle_E\}$

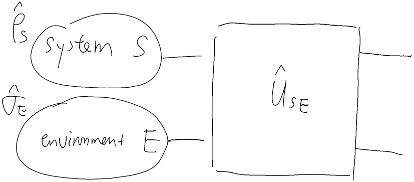
Start from mixed state. $\rho_S = \sum_j \underline{\lambda_j\, |j\rangle_S\langle j|}$ eigenvalues of $\hat{\rho}_S$

dynamics : unitary



$$\hat{U}_{SE}(\hat{\rho}_S \otimes \hat{\sigma}_E)\hat{U}_{SE}^{\dagger}$$

Quantum channel (CPTP maps)



What is the state of the system after evolution?

$$\hat{\rho}_S' = \mathrm{tr}_E\left[\hat{U}_{SE}\left(\hat{\rho}_S \otimes \hat{\sigma}_E\right)\hat{U}_{SE}^{\dagger}\right]$$

observe : ① all operations involved are linear (due to QM)

② it's a map between input state $\hat{\rho}_S \rightarrow \hat{\rho}_S'$

math language : completely positive trace preserving (CPTP)

Quantum channels: coming from unitary evolutions

$$\mathcal{N}(\rho_S) = \mathrm{tr}_E\left\{\hat{U}_{SE}\left(\rho_S \otimes \sigma_E\right)\hat{U}_{SE}^{\dagger}\right\}$$

to further simplify: $\sigma_E = |e_0\rangle\langle e_0|_E$

why pure? ← if not we can purify it and make the environment larger.

then pick a bases $\{|e_k\rangle_E\}$.

$$\mathcal{N}(\rho_S) = \sum_k \langle e_k|\, \hat{U}_{SE}\; \rho_S \otimes |e_0\rangle\langle e_0|\, \hat{U}_{SE}^{\dagger}\, |e_k\rangle$$

$$= \sum_k \left(\langle e_k|\hat{U}_{SE}|e_0\rangle\right) \rho_S \left(\langle e_0|\hat{U}_{SE}^{\dagger}|e_k\rangle\right)$$

$$\mathcal{N}(\rho_S) = \sum_k \langle e_k| \hat{U}_{SE} \; \rho_S \otimes |e_0\rangle\langle e_0| \; \hat{U}_{SE}^\dagger \, |e_k\rangle$$

$$= \sum_k \left(\langle e_k| \hat{U}_{SE} |e_0\rangle\right) \rho_S \left(\langle e_0| \hat{U}_{SE}^\dagger |e_k\rangle\right)$$

$$= \sum_k \hat{L}_k \, \rho_S \, \hat{L}_k^\dagger$$

$\hat{L}_k \equiv \langle e_k| \hat{U}_{SE} |e_0\rangle$ Kraus operators.

completeness. $\sum_k |e_k\rangle\langle e_k|_E = \mathbb{I}_E$.

$$\Rightarrow \sum_k \hat{L}_k^\dagger \hat{L}_k = \sum_k \langle e_0| \hat{U}_{SE}^\dagger |e_k\rangle\langle e_k| \hat{U}_{SE} |e_0\rangle = {}_E\langle e_0| \mathbb{I}_{SE} |e_0\rangle_E = \mathbb{I}_S$$

Examples: Erasure and Depolarizing

$\mathcal{L}_\varepsilon$: Consider a two-level quantum system (a qubit) described by the quantum state $\Psi \in \mathscr{H}$, and consider the "erasure state" $|\varepsilon\rangle$ which lies outside of $\mathscr{H}$ (i.e., $\langle\varepsilon|\Psi|\varepsilon\rangle = 0 \,\forall\, \Psi \in \mathscr{H}$). An erasure channel $\mathcal{L}_\varepsilon$ acts on the qubit as,

$$\mathcal{L}_\varepsilon(\Psi) = (1-\varepsilon)\Psi + \varepsilon\,|\varepsilon\rangle\!\langle\varepsilon|\,,$$

where $0 \leq \varepsilon \leq 1$ is the erasure probability.

Δ_p: Given a qubit Ψ, a depolarizing channel Δ_p acts as follows,

$$\Delta_p(\Psi) = (1-p)\Psi + p\hat{I}/2,$$

where $\hat{I}/2$ is the maximally mixed state and $0 \leq p \leq 4/3$.

unitary extension / isometric extension

$$\mathcal{N}(\rho) = \mathrm{tr}_E\left[U_{SE}(\rho_S \otimes \sigma_E) U_{SE}^{\dagger} \right]$$

$\uparrow$

unitary extension. $U_{SE}^{\dagger} U_{SE} = U_{SE} U_{SE}^{\dagger} = \mathbb{1}_{SE}$.

isometry: lazy version of unitary extension.

$$\hat{V}_{S\to SE} = \sum_k \hat{L}_k \otimes |e_k\rangle$$

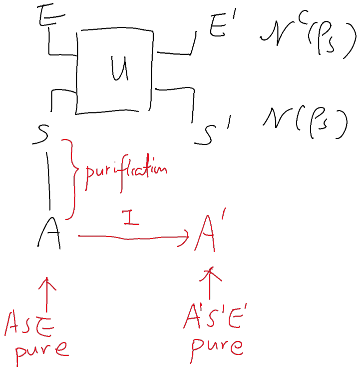
$$\hat{V}_{S\to SE}^{\dagger} \hat{V}_{S\to SE} = \mathbb{1}_S$$

$$V_{S\to SE} V_{S\to SE}^{\dagger} = \mathbb{1}_{SE}$$

$$\mathcal{N}(\rho_S) = \mathrm{tr}_E\left[V_{S\to SE}\, \rho_S\, V_{S\to SE}^{\dagger} \right]$$

Why unitary extension & purification is useful ?

theoretical tool for QIP analyses.



e.g.

Quantum capacity.(single-letter)

$$Q^{(1)}(\mathcal{N}) = \max_{\rho_S} S(\mathcal{N}(\rho_S)) - S(\mathcal{N}^c(\rho_S))$$

$$S(\mathcal{N}^c(\rho_S)) = S(E') = S(A'S') = S(\mathcal{N}\otimes I(\psi_{\rho_S}))$$

Poll 1: Concatenated erasures

Consider two erasure channels $\mathcal{L}_{\varepsilon_1}$ and $\mathcal{L}_{\varepsilon_2}$ where, e.g., $\mathcal{L}_\varepsilon(\Psi) = (1-\varepsilon)\Psi + \varepsilon\, |\varepsilon\rangle\!\langle\varepsilon|$ for some state Ψ. The concatenation of the two erasure channels is another erasure channel, $\mathcal{L}_{\varepsilon_{12}} = \mathcal{L}_{\varepsilon_2} \circ \mathcal{L}_{\varepsilon_1}$. What is the erasure probability ε_{12}? [Hint: The erasure probability is 1 minus the transmission probability.]

(A) $(\varepsilon_1 + \varepsilon_2)/2$

(B) $\varepsilon_1 \varepsilon_2$

(C) $1 - (1-\varepsilon_1)(1-\varepsilon_2)$

Answer 1: Concatenated erasures

Consider two erasure channels $\mathcal{L}_{\varepsilon_1}$ and $\mathcal{L}_{\varepsilon_2}$ where, e.g., $\mathcal{L}_{\varepsilon}(\Psi) = (1-\varepsilon)\Psi + \varepsilon\,|\varepsilon\rangle\langle\varepsilon|$ for some state Ψ. The concatenation of the two erasure channels is another erasure channel, $\mathcal{L}_{\varepsilon_{12}} = \mathcal{L}_{\varepsilon_2} \circ \mathcal{L}_{\varepsilon_1}$. What is the erasure probability ε_{12}? [Hint: The erasure probability is 1 minus the transmission probability.]

(A) $(\varepsilon_1 + \varepsilon_2)/2$

(B) $\varepsilon_1\varepsilon_2$

(C) $1-(1-\varepsilon_1)(1-\varepsilon_2)$

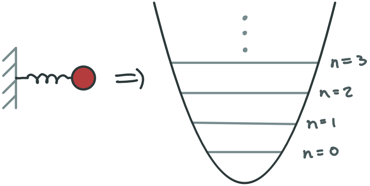
State either gets transmitted or erased. Transmission probability for first channel is $(1-\varepsilon_1)$. Transmission probability for second channel is $(1-\varepsilon_2)$. Total transmission probability is the product of probabilities $(1-\varepsilon_1)(1-\varepsilon_2)$. Erasure probability is thus $1-(1-\varepsilon_1)(1-\varepsilon_2)$.

Center for Quantum Networks NSF-ERC

GAUSSIAN BOSONIC CHANNELS

Recall: Quantum harmonic oscillator

- Free EM field is bosonic field described by harmonic oscillator-like Hamiltonian $\hat{H}_{\text{osc}} = \frac{\hbar\omega}{2}\left(\hat{q}^2 + \hat{p}^2\right)$ with frequency ω



- Canonical position $\hat{q}$ and momentum $\hat{p}$ (quadratures) obey CCR $[\hat{q}, \hat{p}] = i\hat{I}$
- Annihilation operator $\hat{a}$ related via $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$ s.t. $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$

- Equivalently, $\hat{H}_{\text{osc}} = \hbar\omega\hat{n} + \hbar\omega/2$ with number operator $\hat{n} \equiv \hat{a}^\dagger\hat{a}$ and eigenstate (Fock state) $|n\rangle$ where $n \in \mathbb{Z}^+$.
- Explicitly, $|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|\text{vac}\rangle$. Then, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Note $\hat{a}|\text{vac}\rangle = 0$.

Gaussian bosonic channels.

introduce vector of operator for N-mode system:

$$\hat{x} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_N, \hat{p}_N)$$

commutation reation.

$$[\hat{x}_i, \hat{x}_j] = i\Omega_{ij}$$

$$\Omega = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & \\ 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & & \ddots \end{pmatrix}$$

symplectic.
metric

We will follow the previous approach, start from unitary and then to quantum channels for bosonic systems.

Unitary $U(t) = e^{-i\hat{H}t}$ is generated from $\hat{H}$

$$\begin{cases} \text{Gaussian: } \hat{H} \text{ second order in } \hat{p}, \hat{q}. \text{ e.g. } \hat{p}^2, \hat{p}\hat{q}, \hat{q}^2 \\ \text{non-Gaussian } \hat{H} \text{ higher order in } \hat{p}, \hat{q} \text{ e.g. } \hat{p}^3, \hat{q}^3, \dots \end{cases}$$

Gaussian unitary is nice because $\boxed{\hat{U}_{s,d}\overset{\triangle}{\hat{x}}\hat{U}_{s,d} = S\hat{\bar{x}}+d}$

using Hadamard lemma $e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \dots$

linear Hamiltonian: displacement.

$$\hat{H} = i(\alpha \hat{a}^{\dagger} - \alpha^{*} \hat{a}) = \sqrt{2}\,(\mathrm{Re}\,\alpha\, \hat{p} - \mathrm{Im}\,\alpha\, \hat{q})$$

displacement $\hat{D}^{\dagger}(\alpha)\, \hat{a}\, \hat{D}(\alpha) = \hat{a} + \alpha$

$\hat{D}(\alpha)|0\rangle = |\alpha\rangle$, produces coherent state (laser)
when acting on vacuum.

multi-mode $\hat{D}^{\dagger}(\vec{\xi})\, \hat{\vec{x}}\, \hat{D}(\vec{\xi}) = \hat{\vec{x}} + \vec{\xi}$

orthogonal & complete. (analog to Pauli operator)

$$\mathrm{tr}[\hat{D}(\vec{\xi})\, \hat{D}(\vec{\xi}')] = \pi^{N}\, \delta(\vec{\xi} + \vec{\xi}')$$

linear Hamiltonian : displacement.

$$\mathrm{tr}\left[\hat{D}(\vec{\xi})\, \hat{D}(\vec{\xi}') \right] = \pi^N \delta(\vec{\xi} + \vec{\xi}')$$

Wigner characteristic function

$$\chi(\vec{\xi}; \hat{A}) = \mathrm{Tr}\left[\hat{A} \hat{D}(\vec{\xi}) \right]$$

Wigner function

$$W(\vec{x}; \hat{A}) = \int \frac{d^{2N}\xi}{(2\pi)^{2N}} \exp(-i\, \vec{x}^T \Omega \vec{\xi})\, \chi(\vec{\xi}; \hat{A})$$

Gaussian unitary: properties.

$$\hat{U}_{S,d}^{\dagger}\,\hat{x}\,\hat{U}_{S,d} = S\hat{x} + d$$

$$\chi(\xi;\hat{U}_{S,d}\hat{A}\hat{U}_{S,d}^{\dagger}) = \chi(S^{-1}\xi;\hat{A})\,e^{i d^{T}\Omega\xi}$$

$$W(x;\hat{U}_{S,d}\hat{A}\hat{U}_{S,d}^{\dagger}) = W(S^{-1}(x-d);\hat{A})$$

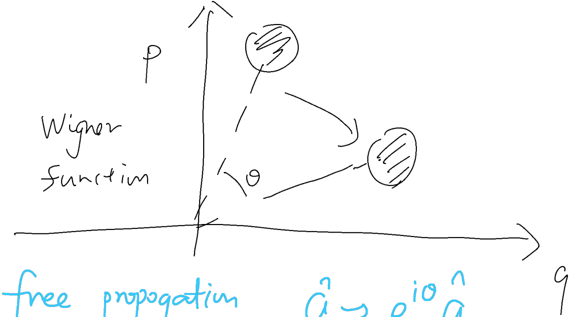
Gaussian unitary are coordinate transforms in phase space.

we omit → for vectors without causing confusion

Gaussian unitary: phase rotation

quadratic (Gaussian): $\hat{H} = \hat{a}^{+}\hat{a}$

phase rotation $\hat{R}(\theta)\,\hat{a}\,\hat{R}(\theta) = e^{-i\theta}\,\hat{a}$



Models free propogation $\hat{a} \rightarrow e^{i\theta}\hat{a}$

Gaussian unitary: single-mode squeezing.

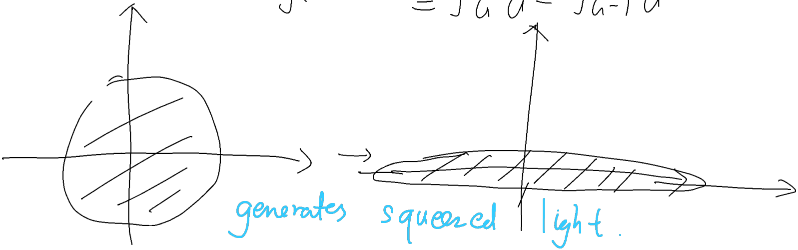
$\hat{H} \propto i(\hat{a}^2 - \hat{a}^{\dagger 2})$

single-mode squeezing $\quad S^\dagger(r)\, a\, S(r) = \cosh r\, \hat{a} - \sinh r\, \hat{a}^\dagger$

sometimes we let $G = (\cosh r)^2$ so that

gain

$$= \sqrt{G}\,\hat{a} - \sqrt{G-1}\,\hat{a}^\dagger$$



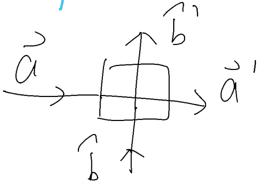
generates squeezed light.

Gaussian unitary: beamsplitter

$$\hat{H} \propto \hat{a}\hat{b}^{+} + \hat{a}^{+}\hat{b}^{-}$$

$$\begin{cases} \hat{a} \to \cos\theta\,\hat{a} + \sin\theta\,\hat{b} \\ \hat{b} \to \cos\theta\,\hat{b} - \sin\theta\,\hat{a} \end{cases}$$

Models beamsplitter.



Gaussian unitary :

$\hat{H} \propto \hat{a}\hat{b} - \hat{a}^{+}\hat{b}^{+}$

two-mode squeezing $\begin{cases} \hat{a} \to \sqrt{G}\,\hat{a} - \sqrt{G-1}\,\hat{b}^{+} \\ \hat{b} \to \sqrt{G}\,\hat{b} - \sqrt{G-1}\,\hat{a}^{+} \end{cases}$

generates two-mode squeezed vacuum.

bosonic version of EPR state

non-Gaussian unitary

higher-order
(non-Gaussian)

$H = \hat{q}^3$. cubic phase gate.

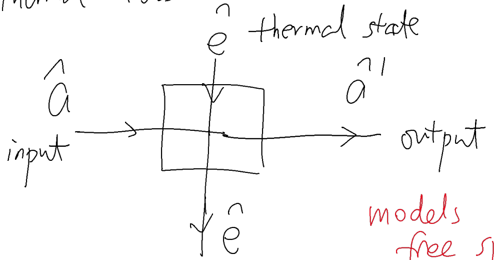
$H = \hat{n}^2$ Kerr nonlinearity.

......

Gaussion channel:

① unitary channel. e.g. displacement channel

② thermal-loss channel.



$$\hat{a}' = \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{e}$$

$$\langle \hat{e}^{+} \hat{e} \rangle = N_B.$$

denote as $\mathcal{L}_{\eta, N_B}$

models light propogation in fiber
free space etc.

Poll 2: Pure loss and erasure

A pure-loss channel $\mathcal{L}_\eta$ has an operator sum representation

$$\mathcal{L}_\eta(\rho) = \sum_{\ell=0}^{\infty} \hat{A}_\ell \rho \hat{A}_\ell^\dagger, \tag{1}$$

with Kraus operators

$$\hat{A}_\ell = \sqrt{\frac{(1-\eta)^\ell}{\ell!}} \eta^{\hat{a}^\dagger \hat{a}/2} \hat{a}^\ell. \tag{2}$$

How many Kraus operators do we need to describe the output $\mathcal{L}_\eta(\rho_1)$ for a single-photon input state ρ_1? [Hint: focus on the $\hat{a}^\ell$ term and recall that $\hat{a}$ annihilates the vacuum.]

(A) 1 (B) 2 (C) 3

Answer 2: Pure loss and erasure

A pure-loss channel $\mathcal{L}_\eta$ has an operator sum representation

$$\mathcal{L}_\eta(\rho) = \sum_{\ell=0}^{\infty} \hat{A}_\ell \rho \hat{A}_\ell^\dagger, \tag{3}$$

with Kraus operators

$$\hat{A}_\ell = \sqrt{\frac{(1-\eta)^\ell}{\ell!}} \eta^{\hat{a}^\dagger \hat{a}/2} \hat{a}^\ell. \tag{4}$$

How many Kraus operators do we need to describe the output $\mathcal{L}_\eta(\rho_1)$ for a single-photon input state ρ_1? [Hint: Focus on the $\hat{a}^\ell$ term and recall that $\hat{a}$ annihilates the vacuum.]

(A) 1 (B) 2 (C) 3

For single-photon state ρ_1 and $\ell \geq 2$, $\hat{a}^\ell \rho_1 \hat{a}^{\ell\,\dagger} = 0$ because $\hat{a}\rho_1\hat{a}^\dagger \propto |\mathrm{vac}\rangle\langle\mathrm{vac}|$ and $\hat{a}\,|\mathrm{vac}\rangle\langle\mathrm{vac}|\,\hat{a}^\dagger = 0$. Thus only first two Kraus operators ($\ell = 0, 1$) are necessary (and given by $\hat{A}_0 = \sqrt{\eta}\hat{I}$ and $\hat{A}_1 = \sqrt{1-\eta}\hat{a}$).

Center for Quantum Networks NSF-ERC

Gassian channel.

thermal-ampifier channel. ← two-mode squeezing.

$$a = \sqrt{G}\,\hat{a} + \sqrt{G-1}\,\hat{e}^{\dagger}$$

$$\langle \hat{e}^{\dagger}\hat{e} \rangle = N_B . \qquad \mathcal{A}_{G,N_B} .$$

additive Gaussian noise (AGN)

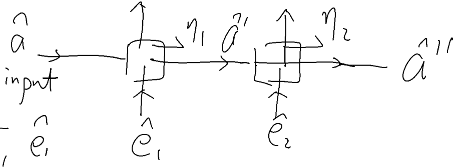
$$\mathcal{N}_{N_B} = \lim_{\eta \to 1} \mathcal{L}_{\eta,\, \frac{N_B}{1-\eta}}$$

mix noise N_B

↗ into the output

Gaussian channel concatenation

$\mathcal{L}_{\eta_2, N_2} \circ \mathcal{L}_{\eta_1, N_1}$:



$$\hat{a}' = \sqrt{\eta_1}\,\hat{a} + \sqrt{1-\eta_1}\,\hat{e}_1$$

$$\hat{a}'' = \sqrt{\eta_2}\,\hat{a}' + \sqrt{1-\eta_2}\,\hat{e}_2 = \cdots = \sqrt{\eta_1\eta_2}\,\hat{a} + \sqrt{1-\eta_1\eta_2}\,\hat{e}_3$$

$$\hat{e}_3 = \frac{\sqrt{\eta_2(1-\eta_1)}\,\hat{e}_1 + \sqrt{1-\eta_2}\,\hat{e}_2}{(1-\eta_1\eta_2)}$$

$$= \mathcal{L}_{\eta_1\eta_2, N_3} \qquad N_3 = \frac{\eta_2(1-\eta_1)N_1 + (1-\eta_2)N_2}{1-\eta_1\eta_2}$$

$$\mathcal{A}_{G,N_2} \circ \mathcal{L}_{\eta,N_1}$$

$$\hat{a}'' = \sqrt{G\eta}\,\hat{a} + \left(\sqrt{G}\sqrt{1-\eta}\,\hat{e}_1 + \sqrt{G-1}\,\hat{e}_2^{\dagger}\right)$$

$$\left\{\begin{array}{llll}
G\eta<1 & \text{overall thermal loss} & & \mathcal{L}_{G\eta,N_3} \\
 & N_3 = \dfrac{G(1-\eta)N_1}{1-G\eta} + \dfrac{G-1}{1-G\eta}(N_2+1) & & \\
G\eta=1. & \text{AGN.} \quad \mathcal{N}_{N_B} & & N_B = (G-1)(N_1+N_2+1) \\
G\eta>1 & \text{thermal amplifier} & & \mathcal{A}_{G\eta,N_4} \\
 & N_4 = \dfrac{G(1-\eta)}{G\eta-1}(N_1+1) + \dfrac{G-1}{G\eta-1}N_2 & &
\end{array}\right.$$

Exercise 1: Amplifier-then-loss is less noisy

Q: From before, we have that $\mathcal{N}_{N_{B_1}} = \mathcal{A}_{G,N_2} \circ \mathcal{L}_{\eta,N_1}$ for $G\eta = 1$, where $N_{B_1} = (G-1)(N_1 + N_2 + 1)$. Show that $\mathcal{N}_{N_{B_2}} = \mathcal{L}_{\eta,N_1} \circ \mathcal{A}_{G,N_2}$ for $G\eta = 1$ and give N_{B_2} explicitly. Prove that $N_{B_2} < N_{B_1}$. Hence, amp-loss is less noisy than loss-amp.

A: Use similar tricks and prove at level of annihilation operators.

$$\hat{a} \xrightarrow{\mathcal{A}_{G,N_2}} \hat{a}' = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}_2^\dagger$$

$$\hat{a}' \xrightarrow{\mathcal{L}_{\eta,N_1}} \hat{a}'' = \sqrt{\eta}\hat{a}' + \sqrt{1-\eta}\hat{e}_1.$$

Then $\hat{a}'' = \sqrt{\eta G}\hat{a} + \sqrt{1-\eta G}\left(\frac{\sqrt{\eta(G-1)}\hat{e}_2^\dagger + \sqrt{1-\eta}\hat{e}_1}{\sqrt{1-\eta G}}\right)$. Equivalent to AGN $\mathcal{N}_{B_2}$ in limit $\eta G \to 1$ with $N_{B_2} = (1-\eta)(N_2 + N_1 + 1)$. Since $1 - \eta = (G-1)/G$ and $(G-1)/G < G - 1$, then $N_{B_2} < N_{B_1}$.

SINGLE PHOTON ENCODINGS

Dual-rail qubit

- Photons have many degrees of freedom (polarization, spatial, angular momentum etc.).
- Each dof can described by set of mode operators $\{\hat{a}_k\}_{k=1}^{M}$ where M is the number of orthogonal modes
- Generally focus on two modes $k \in \{1, 2\}$ to define a photonic qubit. Logical states 0 and 1 are single-photon states

  $$|0\rangle = \hat{a}_1^\dagger |\text{vac}\rangle \quad \text{and} \quad |1\rangle = \hat{a}_2^\dagger |\text{vac}\rangle$$

  s.t. general **dual-rail qubit** $\Psi \in \text{span}\{|0\rangle, |1\rangle\}$
- Technically, $|\text{vac}\rangle = |\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$, $\hat{a}_1^\dagger |\text{vac}\rangle = \hat{a}_1^\dagger \otimes \hat{\mathbb{I}} |\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$ etc.

Single qubit operations

- Single-qubit operations implemented with **passive operations**.
- Passive operations commute with total photon number $\hat{N} = \sum_{k=1}^{2} \hat{a}_k^\dagger \hat{a}_k$
- Consist of unitary beam splitters and phase-shifters, $\hat{U}_{\mathrm{BS}}$ and $\hat{U}_\phi$, with Hamiltonians

$$\hat{H}_{\mathrm{BS}} = i\theta \mathrm{e}^{i\varphi} \hat{a}_1^\dagger \hat{a}_2 + \mathrm{h.c.},$$

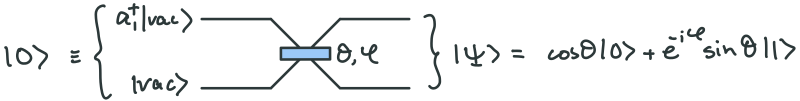
$$\hat{H}_\phi = \sum_{k=1}^{2} \phi_k \hat{a}_k^\dagger \hat{a}_k.$$

Exercise 2: Passive operations

Q: Show that any Hamiltonian of the form $\hat{H} = \sum_{i,j} H_{ij}\hat{a}_i^\dagger \hat{a}_j$ commutes with the total photon number operator $\hat{N} = \sum_{k=1}^{2} \hat{a}_k^\dagger \hat{a}_k$.

A: Equivalent to showing $\sum_k \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_i^\dagger \hat{a}_j\right] = 0$. Use $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\,\hat{B}$ and $[\hat{a}_i^\dagger, \hat{a}_j] = \delta_{ij}$.

Beamsplitter transformation



$$|\Psi\rangle = \cos\theta|0\rangle + e^{-i\varphi}\sin\theta|1\rangle$$

- Action of general **beamsplitter** on mode operators,

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix}}_{\equiv V_{\mathrm{BS}}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

where $V_{\mathrm{BS}}^\dagger V_{\mathrm{BS}} = \mathbb{I}$ and $\det V_{\mathrm{BS}} = 1$.

- Easy to show that

$$|0\rangle \xrightarrow{V_{\mathrm{BS}}} \cos\theta\,|0\rangle + \mathrm{e}^{-i\varphi}\sin\theta\,|1\rangle\,,$$
$$|1\rangle \xrightarrow{V_{\mathrm{BS}}} -\mathrm{e}^{i\varphi}\sin\theta\,|0\rangle + \cos\theta\,|1\rangle\,.$$

Poll 3: Pauli-X with a beam splitter

Consider two input modes $\hat{a}_1$ and $\hat{a}_2$ into a general beam splitter transformation with outputs $\hat{a}_1'$ and $\hat{a}_2'$ given as,

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}.$$

Up to a global phase, can we implement the Pauli-X matrix $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with this transformation?

(A) Yes

(B) No

Answer 3: Pauli-X with a beam splitter

Consider two input modes $\hat{a}_1$ and $\hat{a}_2$ into a general beam splitter transformation with outputs $\hat{a}_1'$ and $\hat{a}_2'$ given as,

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}.$$

Up to a global phase, can we implement the Pauli matrix $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with this transformation?

(A) Yes

(B) No

Choose, e.g., $\theta = \varphi = \pi/2$. Substitute into rotation matrix above to find $\left(\begin{smallmatrix} 0 & \mathrm{e}^{i\pi/2} \\ \mathrm{e}^{i\pi/2} & 0 \end{smallmatrix}\right) \propto X$. This is because $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$, and $-\mathrm{e}^{-i\pi/2} = \mathrm{e}^{i\pi/2}$.

Exercise 3: Hadamard

Q: Transformation matrices for phase shifts and beamsplitter,

$$V_\phi = \begin{pmatrix} \mathrm{e}^{i\phi_1} & 0 \\ 0 & \mathrm{e}^{i\phi_2} \end{pmatrix} \quad \text{and} \quad V_{\mathrm{BS}} = \begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix}.$$

What combination of phase-shifters and beamsplitters produces the Hadamard matrix, $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$?

A: Choose $\phi_1 = 0$, $\phi_2 = \pi$ s.t. $V_\phi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and choose $\theta = \pi/4$ and $\varphi = \pi/2$ s.t. $V_{\mathrm{BS}} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Then $H = V_\phi V_{\mathrm{BS}}$.

Spatial, polarization, and time-bin encodings

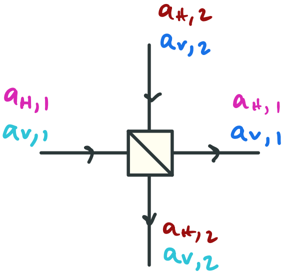
- When choosing photonic dof for encoding, questions to consider:
  - Is the dof easy to manipulate?
  - Is the dof robust to relevant noise sources?
  - If necessary, can we scale-up for quantum information processing with many photons?
- Answers to these questions depend on context.
- Common encodings:
  (i) *Spatial*: Photon with fixed frequency ω, polarization etc., but may traverse two distinct paths $k = 1, 2$. Interaction by overlapping paths at, e.g., beamsplitters. Phase shifts via path lengths s.t. $\phi_k = \omega L_k / c$.
  (ii) *Polarization*: Photon with fiexed frequency, spatial path etc., but may be in a superposition of polarization states. Horizontal H and vertical V polarization define logical states, $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$. Birefringent materials implement single-photon operations.
  (iii) *Time-bin*: Photon with fixed frequency, polarization, spatial path etc., but may occupy two distinct time-binned intervals $k = e, l$ (e for early, l for late). Fast optical switches and delays implement single-photon operations.

Swapping encodings: Polarization to spatial

- Swapping between encodings is possible
- E.g., given two polarization modes H, V and two spatial modes $1, 2$, implement a polarizing beamsplitter (PBS) s.t.

$$\hat{a}_{H,1} \to \hat{a}_{H,1} \quad \text{and} \quad \hat{a}_{H,2} \to \hat{a}_{H,2},$$
$$\hat{a}_{V,1} \to \hat{a}_{V,2} \quad \text{and} \quad \hat{a}_{V,2} \to \hat{a}_{V,1}.$$

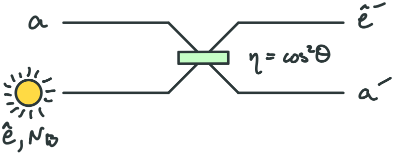
- H gets transmitted while V gets reflected. Follow up by a polarization rotation results in swap from polarization qubit to spatial qubit



SINGLE PHOTON EVOLUTION

Single-photon evolution through thermal loss channel

- Most communication links (i.e., quantum channels) are over noisy fibers or free-space links, which can be accurately described by thermal loss channels
- Focus on the action of a thermal loss channel $\mathcal{L}_{\eta,N_B}$ on a single-photon state ρ_1
- Physically, background quanta N_B can be the population of the environment—originating from, e.g., the sun, the moon, or background lights for free-space links—whereas the loss probability $1-\eta$ of the channel is equal to the absorption probability of the medium.
- E.g., given a fiber of length L, $\eta = \mathrm{e}^{-\alpha L}$ where α is an attenuation coefficient (typically quoted in dB/km). The exponential attenuation is a consequence of the Beer-Lambert law for absorptive media.



Thermal loss: Channel decomposition

- Consider a thermal-loss channel $\mathcal{L}_{\eta,N_B}$ which has the following decomposition $\mathcal{L}_{\eta,N_B} = \mathcal{A}_{G,0} \circ \mathcal{L}_{\tau,0}$ with
$$\tau G = \eta \quad \text{and} \quad \frac{G-1}{1-G\tau} = N_B.$$
- Parameters τ and G are related to η and N_B via
$$G = (1-\eta)N_B + 1 \quad \text{and} \quad \tau = \frac{\eta}{(1-\eta)N_B + 1}.$$
- To show decomposition:

Thermal loss: Operator-sum representation

- Consider Kraus operators $\{\hat{A}_\ell\}_{\ell=0}^{\infty}$ of pure-loss channel $\mathcal{L}_{\tau,0}$

$$\hat{A}_\ell = \sqrt{\frac{(1-\tau)^\ell}{\ell!}}\tau^{\hat{a}^\dagger \hat{a}/2}\hat{a}^\ell.$$

- Consider Kraus operators $\{\hat{B}_k\}_{k=0}^{\infty}$ of quantum-limited amplifier $\mathcal{A}_{G,0}$,

$$\hat{B}_k = \sqrt{\frac{1}{k!}\frac{1}{G}\left(\frac{G-1}{G}\right)^k}\hat{a}^{\dagger k}G^{-\hat{a}^\dagger \hat{a}/2}.$$

- Using $\mathcal{L}_{\eta,N_B} = \mathcal{A}_{G,0} \circ \mathcal{L}_{\tau,0}$, thermal loss channel then has

$$\mathcal{L}_{\eta,N_B}(\rho) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\hat{B}_k\hat{A}_\ell\rho\hat{A}_\ell^\dagger\hat{B}_k^\dagger.$$

Center for Quantum Networks NSF-ERC

Thermal loss: single-photon input

$\mathcal{L}_{\eta,N_B}(\rho_1) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \hat{B}_k \hat{A}_\ell \rho_1 \hat{A}_\ell^\dagger \hat{B}_k^\dagger$

- Consider single-photon input ρ_1 with output $\mathcal{L}_{\eta,N_B}(\rho_1)$.
- Terms $\hat{A}_\ell \rho_1 \hat{A}_\ell^\dagger$ are only non-zero when $\ell = 0, 1$. Thus,

$$\hat{A}_0 \rho_1 \hat{A}_0^\dagger = \tau \rho_1 \quad \text{and} \quad \hat{A}_1 \rho_1 \hat{A}_1^\dagger = (1-\tau)\left|\mathrm{vac}\rangle\!\langle \mathrm{vac}\right|.$$

- With probability τ, the photon is transmitted. With probability $1-\tau$, the photon is lost.

Poll 4: Pure loss and erasure cont.

When acting on a single-photon state ρ_1, the pure-loss channel $\mathcal{L}_\tau$ is equivalent to an erasure channel $\mathcal{L}_\varepsilon$ with erasure probability $\varepsilon = 1 - \tau$. What is the erasure state in this case? [Hint: Note that we are *losing* photons via loss.]

(A) Vacuum state

(B) Completely mixed single-photon state

(C) State with ≥ 2 photons

Answer 4: Pure loss and erasure cont.

When acting on a single-photon state ρ_1, the pure-loss channel $\mathcal{L}_\eta$ is equivalent to an erasure channel $\mathcal{L}_\varepsilon$ with erasure probability $\varepsilon = 1 - \eta$. What is the erasure state in this case? [Hint: Note that we are *losing* photons via loss.]

(A) Vacuum state

(B) Completely mixed single-photon state

(C) State with ≥ 2 photons

Explicitly, $\mathcal{L}_\tau(\rho_1) = \tau\rho_1 + (1-\tau)\,|\mathrm{vac}\rangle\langle\mathrm{vac}|$. With probability τ, the photon is transmitted, and with probability $1-\tau$, the photon is lost.

Thermal loss: single-photon input cont.

$\mathcal{L}_{\eta,N_B}(\rho_1) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \hat{B}_k \hat{A}_\ell \rho_1 \hat{A}_\ell^\dagger \hat{B}_k^\dagger$

- More complicated for amplifier $\mathcal{A}_{G,0}$ due to adding photons
- Relevent operators are $\hat{B}_k$ for $k = 0, 1$,

$$\hat{B}_0 = \sqrt{\frac{1}{G}} G^{-\hat{a}^\dagger \hat{a}/2} \quad \text{and} \quad \hat{B}_1 = \sqrt{\frac{1}{G}\left(\frac{G-1}{G}\right)} \hat{a}^\dagger G^{-\hat{a}^\dagger \hat{a}/2}.$$

- Appending to pure-loss channel leads to,
  (1) $\hat{B}_0 \hat{A}_0 \rho_1 \hat{A}_0^\dagger \hat{B}_0 = \frac{\tau}{G^2} \rho_1$; photon is unaffected by the channel
  (2) $\hat{B}_1 \hat{A}_0 \rho_1 \hat{A}_0^\dagger \hat{B}_1 = \frac{2(G-1)}{G^3} \tau \rho_2$; one noisy photon added to state.
  (3) $\hat{B}_0 \hat{A}_1 \rho_1 \hat{A}_1^\dagger \hat{B}_0 = \frac{(1-\tau)}{G} |\mathrm{vac}\rangle\langle\mathrm{vac}|$; photon is just lost.
  (4) $\hat{B}_1 \hat{A}_1 \rho_1 \hat{A}_1^\dagger \hat{B}_1 = \frac{G-1}{G^2} (1-\tau) \Theta_1$; photon is lost and replaced with a single noisy photon state Θ_1. [$\Theta_1 = \hat{I}/2$ for completely mixed photonic qubit.]

Thermal loss: single-photon input cont.

$\mathcal{L}_{\eta,N_B}(\rho_1) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\hat{B}_k\hat{A}_\ell\rho_1\hat{A}_\ell^\dagger\hat{B}_k^\dagger$

- Overall

$$\mathcal{L}_{\eta,N_B}(\rho_1) = \frac{\tau}{G^2}\rho_1 + \frac{G-1}{G^2}(1-\tau)\Theta_1 + \frac{(1-\tau)}{G}\,|\mathrm{vac}\rangle\langle\mathrm{vac}| + \frac{(G-1)^2+2\tau(G-1)}{G^2}\rho_{\geq 2\text{ photons}},$$

where $\rho_{\geq 2\text{ photons}}$ is a quantum state with more than two photons.

- Transmission event probabilities,

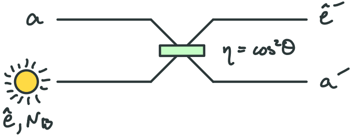
$$p_{\mathrm{success}} = \frac{\tau}{G^2} = \frac{\eta}{[(1-\eta)N_B+1]^3} \quad \text{(successful transmission)}$$

$$p_{\Theta_1} = \frac{G-1}{G^2}(1-\tau) = \frac{(1-\eta)^2N_B(N_B+1)}{[(1-\eta)N_B+1]^3} \quad \text{(random photon)}$$

$$p_{\mathrm{vac}} = \frac{(1-\tau)}{G} = \frac{(1-\eta)(N_B+1)}{[(1-\eta)N_B+1]^2} \quad \text{(receive nothing)}$$

$$p_{\geq 2} = 1 - p_{\mathrm{success}} - p_{\mathrm{depolarizing}} - p_{\mathrm{vac}} \quad \text{(receive} \geq 2 \text{ photons)}$$

Exercise 4: Low thermal noise



Q: Assume $N_B \ll 1$. Expand p_{success}, p_{Θ_1}, and p_{vac} to first order in N_B. Show that $p_{\geq 2} = 2\eta(1-\eta)N_B + \mathcal{O}\left(N_B^2\right)$. Can you intuitively explain result?

A: Expanding previous expressions,

$$p_{\text{success}} = \frac{\eta}{\left[(1-\eta)N_B + 1\right]^3} \approx \eta\left(1 - 3N_B(1-\eta)\right) + \mathcal{O}(N_B)^2$$

$$p_{\Theta_1} = \frac{(1-\eta)^2 N_B(N_B+1)}{\left[(1-\eta)N_B + 1\right]^3} \approx (1-\eta)^2 N_B + \mathcal{O}\left(N_B^2\right)$$

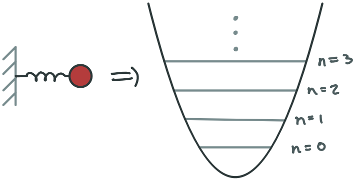
$$p_{\text{vac}} = \frac{(1-\eta)(N_B+1)}{\left[(1-\eta)N_B + 1\right]^2} \approx (1-\eta)\left(1 - (1-2\eta)N_B\right) + \mathcal{O}\left(N_B^2\right).$$

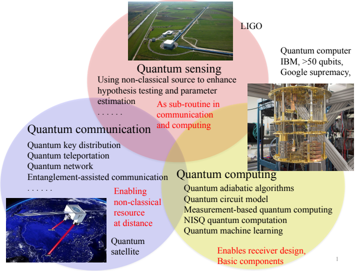
Then, $p_{\geq 2} = 1 - p_{\text{success}} - p_{\Theta_1} - p_{\text{vac}} \approx 2\eta(1-\eta)N_B + \mathcal{O}\left(N_B^2\right)$

RECAP AND EXIT SURVEY

Course recap

- Briefly discussed how photons are good for just about anything (sensing, computing, communication)
- Reviewed general description of evolution and quantum channels (Kraus operators, purification, unitary extension)
- Analyzed Gaussian bosonic channels (loss, amplifier, AGN) and their influence at single-photon level
- Surveyed single-photon encodings and single-qubit operations (e.g., passive operations on dual-rail qubit)




LIGO

Quantum sensing

Using non-classical source to enhance hypothesis testing and parameter estimation

. . . . . .

As sub-routine in communication and computing

Quantum computer IBM, >50 qubits, Google supremacy,

Quantum communication

Quantum key distribution
Quantum teleportation
Quantum network
Entanglement-assisted communication
. . . . . .

Enabling non-classical resource at distance

Quantum satellite

Quantum computing

Quantum adiabatic algorithms
Quantum circuit model
Measurement-based quantum computing
NISQ quantum computation
Quantum machine learning

Enables receiver design, Basic components

Center for
Quantum Networks
NSF Engineering Research Center

Course Evaluation Survey

We value your feedback on all aspects of this short course. Please go to the link provided in the Zoom Chat or in the email you will soon receive to give your opinions of what worked and what could be improved.

CQN Winter School on Quantum Networks

Funded by National Science Foundation Grant #1941583