Information in a Photon

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Quantum information processing

• Communications
  – Deep-space lasercom
  – Quantum security: QKD, covert communications

• Computing
  – Factoring [breaks RSA, Diffie-Hellman]
  – Search and optimization
  – Blind quantum computing
  – Multiparty privacy-preserving computations
  – Simulations of complex quantum systems

• Sensing
  – Passive imaging (astronomy, microscopy, SDA)
  – Active photonic sensors (RF photonic antennas, fiber-optic gyroscopes, beam deflectometers, surface topography, endoscopy, lidars, …)
  – Gravitational wave sensors (LIGO – squeezed light)
  – Magnetic field sensing (Neuronal imaging, chip testing)
• Wherever light gets used in encoding, extracting, carrying or processing information [communications, sensing, imaging, computing, simulations, …], what is the best performance permissible by the laws of quantum physics?
Quantum optics meets Information theory

Quantum optics – quantum theory of light and its interaction with matter
- Non-linear optics
- Atomic systems
- Many-body physics

Information theory – quantifying “information” in the context of communications, sensing and computing
- Detection theory
- Estimation theory
- Data compression
- Channel coding
Course objective

• Introduction to some of the mathematical tools necessary to understand the quantum representation of classical laser light, and certain non-classical states of light, including single-photon and multi-photon states and squeezed states of light

• Using these tools to understand how to manipulate (classical and quantum) light using interference

• Using basic tools from quantum information theory to uncover applications of quantum techniques (sources of light, ways to manipulate light, and detection schemes) for improved performance in encoding, transmission and processing of information
Course plan

• **Module 1**: Quantum limits of information encoded in laser light [Saikat Guha]

• **Module 2**: Quantum information advantage arising from interfering photons [Christos Gagatsos]

• Multiple choice questions interspersed through the course, to be answered through zoom polls

• Post-lecture survey
Module 1: Quantum limits of information encoded in laser light
Digital (classical) communications

Information source → Transduction → Sampling → Quantization

Electrical (voltage or current) signal

Transduction (analog signal)

Sampling

Quantization

Error correction encoder

Modulation

Modulated codewords: Laser pulses (binary phase shift keying modulation format shown)

Noisy channel

Error correction decoder

Noisy / attenuated modulated codeword

Optical process → Detection → Electrical signal process

Receiver

Interpolation → Filter → Transduction

(analog signal + tolerable noise)

Code rate $R = \frac{k}{n}$

$k=3$

$n=6$

$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \end{pmatrix}$
“Quantum limit” of an optical receiver

Optical receiver is a “mini quantum computer”

 Rengaswamy, Seshadreesan, SG, Pfister, Nature npj Quantum Inf 7, 97 (2021)  
Random process describing **Discrete** events in **continuous** time and/or space

- Where such random processes might occur
  - Spiking patterns in neurons, bank teller queue, photon detection, …
- Arrival process, $I(t)$; counting process, $N(t)$
  - $I(t)$: random arrivals (each arrival denoted by a delta function)
  - $N(t)$: number of occurrences (arrivals) before time $t$,
  \[ N(t) = \int_0^t I(t) dt \]

Uptick diagram:

- $I(t)$
- $N(t)$

Arrival rate: $\lambda(t)$

Mean # arrivals:
\[ N = \int_0^T \lambda(t) dt \]

Constant rate:
\[ \lambda(t) = \lambda \]
\[ N = \lambda T \]
Poisson point process (PPP)

1. Probability of an arrival in a tiny time step is equal to the rate of the arrival process (at that time) times the size of the time increment.

2. Probabilities of arrivals in disjoint time steps are statistically independent.

Arrival rate: $\lambda(t)$
Mean # arrivals:
$$N = \int_0^T \lambda(t) dt$$
Constant rate:
$$\lambda(t) = \lambda$$
$$N = \lambda T$$
Probability distribution of inter-arrival time

\[ I(t) \]

\[ 0 \quad t_1 \quad t_2 \quad t_3 \quad \cdots \quad t_k \quad T \quad t \]

\[ p = \lambda \Delta t \]

Arrival rate: \( \lambda(t) = \lambda \)

Time of first arrival

Probability distribution of the time of first arrival, \( P_{\tau}(\tau), \tau \geq 0 \)

Probability distribution of the total number of arrivals, \( P_K[k], k = 0, 1, \ldots \)
Inter-arrival times and number of arrivals

Let us denote by $t$, the time of first arrival; $P[t = \tau] = (1 - p)^{m-1}p;\ m = 1, 2, \ldots$

c.d.f., $F_T(\tau) = P[t \leq \tau] = \sum_{j=1}^{m} (1 - p)^{j-1}p = 1 - (1 - p)^m$

$p.d.f., P_T(\tau) = \frac{d}{d\tau} F_T(\tau) = \lambda e^{-\lambda \tau};\ \tau \geq 0$

Probability distribution of the total number of arrivals $K$, $P_K[k] = \binom{n}{k} p^k (1 - p)^{n-k}$

Probability of one arrival in a $\Delta t$ interval, $p = \lambda \Delta t$

$\tau = m\Delta t$  

$N = \lambda T$  

$T = n\Delta t$
Distribution of the number of arrivals

\[ \tau = m \Delta t \quad T = n \Delta t \]
\[ p = \lambda \Delta t \quad N = \lambda T \]

Probability distribution of the total number of arrivals \( K \), \( P_K[k] = \binom{n}{k} p^k (1 - p)^{n-k} \)

**Problem 1:** What is the distribution of the total number of arrivals \( K \)? \( (k=0, 1, 2, \ldots) \)

A: \[ P_K[k] = (1 - 1/N)^k (1/N) \]
B: \[ P_K[k] = e^{-N} N^k / k! \]
C: \[ P_K[k] = N^{k^2} - N^{(k+1)^2} \]
D: I do not know.
A pulse of laser light

• Quasi-mono-chromatic laser light pulse: in $\sqrt{\text{photons/m}^2\text{-sec}}$ units

$$\tilde{E}(\mathbf{r}, t) = E(\mathbf{r}, t)e^{-j\omega_0 t + \phi}, \quad t \in (0, T], \quad \mathbf{r} \in \mathcal{A}$$

Spatial and temporal dependence may not be factorable in general

= $\psi(\mathbf{r}) s(t) e^{-j\omega_0 t + \phi}, \quad \mathbf{r} \equiv (x, y)$

• Mean photon number, $N_S = \int_0^T \int_{\mathcal{A}} |\tilde{E}(\mathbf{r}, t)|^2 d\mathbf{r} dt$

$$\alpha = \sqrt{N_S} e^{j\phi}$$

Phase space picture: once we identify a spatio-temporal-polarization mode, a complex number describes the state of the laser pulse

No detector can accurately measure the field $E(\mathbf{r}, t)$
Mode (space / time / polarization)

• An optical mode is the “shape” of a confined EM field in space, time and polarization (the three independent degrees of freedom of the photon)

• Time & Frequency are the same degree of freedom (related by Fourier transform)

\[ \phi_\nu(\mathbf{r}, t); \mathbf{r} \in \mathcal{A}, t \in [0, T), \nu = 1, 2 \]

\[
\int_\mathcal{A} \int_0^T \phi_\nu(\mathbf{r}, t) \phi_\nu^*(\mathbf{r}, t) \, d\mathbf{r} \, dt \\
= \int_\mathcal{A} \int_0^T |\phi_\nu(\mathbf{r}, t)|^2 \, d\mathbf{r} \, dt = 1
\]

We will take a mode to be normalized
Orthogonal modes

- Two modes \( \phi_\nu(r, t) \) and \( \psi_\mu(r, t) \) are orthogonal if,

\[
\int_A \int_0^T \phi_\nu(r, t)\psi^*_\mu(r, t)\,dr\,dt = 0
\]

- If \( \nu \neq \mu \), the two modes will be orthogonal regardless of their spatial and temporal shapes.
- When \( \nu = \mu \), we will drop the polarization subscript.
- When we say two “temporal modes” \( s_1(t) \) and \( s_2(t) \) are orthogonal, we will implicitly assume the mode functions being referred to have the same spatial modes and polarization.

\[
E(t) = a_1\phi_1(t) + a_2\phi_2(t) + 0\phi_3(t) + \ldots
\]
Maximum number of orthogonal modes

• Consider temporal modes, $\phi_k(t), \ k = 1, \ldots K$
  \[
  \Phi_k(f) = \int \phi_k(t) e^{-2\pi j ft} dt
  \]
  \[
  \phi_k(t) = \int \Phi_k(f) e^{2\pi j ft} df
  \]

• …and their Fourier transforms,

• How many (K) orthogonal modes $\phi_k(t)$ can be “fit into” a time-bandwidth product of $T \times W$? i.e.,
  \begin{align*}
  - \quad & \phi_k(t) = 0, \ t \notin [0, T], \ \text{and} \ \Phi_k(f) = 0, \ f - f_0 > \frac{W}{2} \\
  - \quad & \text{While ensuring orthogonality:} \ \int_0^T \phi_k(t) \phi_l^*(t) dt = \delta_{kl}
  \end{align*}

• Answer: $K \approx WT$, and these optimal mode functions are “Prolate Spheroidal” functions

• All of above holds for spatial modes as well

Some intuition: choices of WT modes

Each mode fills up the same BW, but different time extents within $[0, T)$

Each mode fills up the same time $[0, T)$, but different freq extents within $[-W/2, W/2]$
Example: flat-top temporal mode

\[ \phi(t) = \begin{cases} \frac{1}{\sqrt{T}}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases} \]

Coherent state of this mode: \( |\alpha\rangle \)

\[ a, \quad \frac{1}{\sqrt{T}}, \quad a\phi(t)e^{j(2\pi f_0 t + \theta)} \]

Phase-space representation:

\[ \alpha = \sqrt{N} e^{j\theta} \]

\[ \sqrt{N} = a \]
Photon detection on a coherent state of the flat-top mode (a “square pulse”)

- PPP of constant rate

\[ s(t) = E e^{j(\omega_0 t + \phi)} \]

Mean photon number in pulse, 
\[ N = \int_0^T \lambda(t) \, dt = \lambda T = E^2 T \]
On-off keying (OOK) modulation

\[ H_1 \quad |0\rangle \]

\[
\begin{align*}
0 & \quad \lambda_1 = 0 \\
N_1 & = 0
\end{align*}
\]

\[ T \]

assumes equal priors:

\[ P(H_1) = P(H_2) = \frac{1}{2} \]

“Maximum Likelihood” decision rule:

\[ k = 0 \text{ click } \square \text{ say } H_1 \]

\[ k > 0 \text{ clicks } \square \text{ say } H_2 \]

\[ P_e = P(H_1)P(H_2|H_1) + P(H_2)P(H_1|H_2) \]

\[ = \frac{1}{2} P(k > 0|H_1) + \frac{1}{2} \times P(k = 0|H_2) \]

Probability of > 0 photon arrivals if \( H_1 \) is actually true (\( N = 0 \) photon pulse incident on detector)

Probability of 0 photon arrivals if \( H_2 \) is actually true (\( N \) photon pulse incident on detector)

\[ = \frac{1}{2} \times 0 + \frac{1}{2} \times e^{-N} \]

Recall the Poisson distribution, evaluate at \( k=0 \):

\[ P_K[k] = \frac{e^{-N}N^k}{k!}, \, k \in \mathbb{Z} \]
Binary phase shift keying (BPSK)

\[ E e^{j\phi} \text{ with, } \phi = \{0, \pi\} \]

Mean photon number in the pulse is the same for either state:

\[ N = |\alpha|^2 = E^2T \]
Interference (passive linear optics)

- **Beamsplitter**
  \[
  |\beta_2\rangle = \eta |\beta_1\rangle + (1-\eta) |\alpha_1\rangle
  \]
  \[
  \begin{bmatrix}
  \beta_1 \\
  \beta_2
  \end{bmatrix}
  = U
  \begin{bmatrix}
  \alpha_1 \\
  \alpha_2
  \end{bmatrix}
  \]
  Transmissivity, \( \eta = \cos^2 \theta \)
  Phase, \( \phi \in (0, 2\pi) \)

  \[
  U(\theta, \phi) = \begin{pmatrix}
  \cos \theta & e^{i\phi} \sin \theta \\
  \sin \theta & -e^{i\phi} \cos \theta
  \end{pmatrix}
  \]

- **Examples**

  **Destructive interference**
  \[
  |\sqrt{2\alpha}\rangle
  \]
  \[
  \eta = \sqrt{1/2}, \phi = 0
  \]

  **Pure phase**
  \[
  |\alpha\rangle
  \]
  \[
  \eta = 0
  \]

  **Nulling (displacement)**
  \[
  |\alpha\rangle \approx |\alpha + \beta\rangle
  \]
  \[
  \eta \approx 1, \phi = 0
  \]
  \[
  |\beta\rangle \approx \frac{|\beta\rangle}{\sqrt{1 - \eta}}
  \]
Mach-Zehnder interferometer (MZI)

\[
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} = U \begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix}
\]

\[
U(\theta, \phi) = \begin{pmatrix}
\cos \theta & e^{i \phi} \sin \theta \\
\sin \theta & -e^{i \phi} \cos \theta
\end{pmatrix}
\]
Arbitrary N-mode linear optical unitary

- Any N-by-N unitary $U$ can be realized with $M = \frac{N(N-1)}{2}$ MZIs. Therefore, one needs tuning $N(N-1)$ phases to realize any $U$.

\[
\begin{align*}
\beta &= U\alpha \\
|\alpha_1\rangle &\quad |\beta_1\rangle \\
\vdots &\quad \vdots \\
|\alpha_n\rangle &\quad |\beta_n\rangle
\end{align*}
\]

Reck et al., PRL 73, 1 (1994)
Clements et al., Optica 3 (12), 1460-1465 (2016)
On-off keying (OOK), Kennedy receiver

\[ |0\rangle \quad \text{H}_1 \]
\[ \lambda_1 = 0 \quad N_1 = 0 \]
\[ \eta \approx 1 \]
\[ \frac{\beta}{\sqrt{1 - \eta}} \]

\[ |\gamma\rangle \xrightarrow{\text{H}_1} |\gamma + \beta\rangle \]

\[ \text{H}_2 \]
\[ \lambda_2 = E_0^2 \quad N_2 = E_0^2 T = |\alpha|^2 = N \]

\[ |\alpha\rangle \]

\[ E_0 + E_1 \quad \text{H}_1 \]
\[ \lambda_1 = E_1^2 \quad N_1 = E_1^2 T = \beta^2 \]

\[ \text{H}_2 \]
\[ \lambda_2 = (E_0 + E_1)^2 \quad N_2 = (E_0 + E_1)^2 T = (\alpha + \beta)^2 \]
P(error) for the Kennedy receiver

- Let us use the same decision rule
  - click = “on”, no-click = “off”

\[
P_e = \frac{1}{2} P(H_1) P(H_2|H_1) + \frac{1}{2} P(H_2) P(H_1|H_2)
\]

\[
= \frac{1}{2} P(k > 0|H_1) + \frac{1}{2} \times P(k = 0|H_2)
\]

\[
= \frac{1}{2} \left( 1 - e^{-\beta^2} \right) + \frac{1}{2} e^{-(\alpha + \beta)^2}
\]

Optimize (minimize) this over the choice of $\beta$
Discriminating BPSK pulses \( \{| - \alpha \rangle, |\alpha \rangle \} \)

- Kennedy (requires perfect **amplitude and phase** reference)
  - Exact nulling \( \{| - \alpha \rangle, |\alpha \rangle \} \rightarrow \{|0 \rangle, |2\alpha \rangle \}, |\alpha|^2 = N \)
    
    R. Kennedy, MIT Research Laboratory for Electronics, Quarterly Progress Report 110, 219 – 225 (1972)
  
  - Optimized nulling \( \{| - \alpha \rangle, |\alpha \rangle \} \rightarrow \{|\beta \rangle, |2\alpha + \beta \rangle \} \)
    

\[
P_e(N) = \min_\beta \left[ \frac{1}{2} e^{-(2\alpha + \beta)^2} + \frac{1}{2} \left( 1 - e^{-\beta^2} \right) \right]
\]

\[
= \frac{1}{2} e^{-4N}, \ \beta = 0 \ \text{ (exact nulling, suboptimal)}
\]

Can we build a receiver for BPSK that does NOT require an amplitude reference?
Homodyne detection

\[ N = E^2T = \alpha^2 \]

Assume for now that both \( \alpha, \alpha_{LO} \) are real, and \( N_{LO} \gg N \)

\[ Y = qGS(K_+ - K_-) \]

\[ K_+ \sim \text{Poisson}(N_+) \sim \mathcal{N}(N_+, N_+); \quad N_+ = \left| \frac{\alpha + \alpha_{LO}}{\sqrt{2}} \right|^2 \]

\[ K_- \sim \text{Poisson}(N_-) \sim \mathcal{N}(N_-, N_-); \quad N_- = \left| \frac{\alpha - \alpha_{LO}}{\sqrt{2}} \right|^2 \]

\[ Y \sim \mathcal{N}(\mu, \sigma^2) \]

\[ \mu = qGS(N_+ - N_-) = 2qGS\alpha\alpha_{LO} \]

\[ \sigma^2 = (qGS)^2(N_+ + N_-) = (qGS)^2(\alpha^2 + \alpha_{LO}^2) \]

Poisson(\( \lambda \)) \approx Gaussian(\( \lambda, \lambda \)), \( \lambda \gg 1 \)

\[ e^{-\lambda} \frac{\lambda^r}{r!} \approx \frac{1}{\sqrt{2\pi \lambda}} e^{-\frac{(r-\lambda)^2}{2\lambda}} \]
Homodyne detection: output distribution

\[ Y \sim \mathcal{N}(\mu, \sigma^2) \]

\[ \mu = qGS(N_+ - N_-) = 2qGS\alpha\alpha_{LO} \]

\[ \sigma^2 = (qGS)^2(N_+ + N_-) = (qGS)^2(\alpha^2 + \alpha_{LO}^2) \]

Pick the scaling constant \( S \) such that the mean \( \mu = \alpha \)

**Problem 2:** What is the distribution of \( Y \)?

A: \( Y \sim \mathcal{N}(\alpha, 1/4) \)

B: \( Y \sim \mathcal{N}(\alpha, 1/2) \)

C: \( Y \sim \mathcal{N}(\alpha, 1) \)

D: I do not know.
Break [5 minutes]
Homodyne detection (continued)

- Local Oscillator (LO) has a phase offset with the input pulse

\[ |\alpha\rangle \]
\[ |\alpha_{LO}\rangle \]
\[ 1/2 \]
\[ i(t) \]
\[ i_+(t) \]
\[ i_-(t) \]
\[ S \int_0^T i(t) dt \rightarrow Y \]

\[ \alpha = \sqrt{N} e^{j\theta} \]
\[ \alpha_{LO} = \sqrt{N_{LO}} e^{j\theta_{LO}} \]
\[ N_{LO} \gg N \]

Show that:

\[ Y \sim \mathcal{N} \left( \text{Re}(\alpha e^{j\theta_{LO}}), \frac{1}{4} \right) \]

i.e.,

\[ Y \sim \mathcal{N} \left( \sqrt{N} \cos(\theta + \theta_{LO}), \frac{1}{4} \right) \]
Homodyne detection

Black box description:

\[ |\alpha\rangle \rightarrow \theta_{\text{LO}} \rightarrow Y \sim \mathcal{N}\left(\Re(\alpha e^{i\theta_{\text{LO}}}), \frac{1}{4}\right) \]

\[ \alpha = \sqrt{N} e^{i\theta} \]

\[ Y \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow p_Y(y) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \]

Gaussian probability distribution

\[ \theta = 0, \quad \theta_{\text{LO}} = 0 \]

\[ -\alpha \quad 0 \quad \alpha \]

\[ P_e = \frac{1}{2} \text{erfc}(\sqrt{2N}) \]
Unambiguous BPSK state discrimination

\[ |\alpha|^2 = N \]

\[ X = 1 \quad |\alpha\rangle \]

\[ X = 2 \quad |\alpha\rangle \]

\[ X = 2 \quad |\alpha\rangle \]

\[ Y = 1 \quad |\alpha\rangle \]

\[ Y = 2 \quad |\alpha\rangle \]

\[ Y = 3 \quad |\alpha\rangle \]

\[ S := p_{Y|X}(y|x) = \begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3
\end{pmatrix} \]

If forced to make a decision, i.e. by mapping the erasure to one possible input, then the average probability of error (assume equal priors), \( P_{e,\text{USD}} = e^{-2N} \)
Recall black box description:

\[ |\alpha\rangle \rightarrow \theta_{LO} \rightarrow Y \sim \mathcal{N}\left(\text{Re}(\alpha e^{j\theta_{LO}}), \frac{1}{4}\right) \]

\[ \alpha = \sqrt{N} e^{j\theta} \]

\[ |\alpha\rangle \rightarrow \theta_{LO} \rightarrow Y_1 \sim \mathcal{N}\left(\text{Re}\left(\frac{\alpha}{\sqrt{2}} e^{j\theta_{LO}}\right), \frac{1}{4}\right) \]

\[ \alpha = \sqrt{N} e^{j\theta} \]

\[ |\alpha\rangle \rightarrow \theta_{LO} \rightarrow Y_2 \sim \mathcal{N}\left(\text{Re}\left(\frac{\alpha}{\sqrt{2}} e^{j(\theta_{LO} - \frac{\pi}{2})}\right), \frac{1}{4}\right) \]

Define: \( X_1 = Y_1 \sqrt{2} \), \( X_2 = Y_2 \sqrt{2} \)

We can show: \( X_1 \sim \mathcal{N}\left(\text{Re}(\alpha e^{j\theta_{LO}}), \frac{1}{2}\right) \)

\[ X_2 \sim \mathcal{N}\left(\text{Im}(\alpha e^{j\theta_{LO}}), \frac{1}{2}\right) \]
BPSK discrimination performance

\[ \{ |\alpha\rangle, | - \alpha\rangle \}, |\alpha|^2 = N \]

\[ P_e,\text{min} = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-4N}} \right] \]

- Pink: USD forced to make hard decision
- Red: Homodyne
- Blue: Kennedy
- Black: Optimal (Dolinar)

\[ \sim e^{-4N} \text{ vs. } \sim e^{-2N} \]
Quantum states and measurements

- A (pure) state is described by a unit-norm column vector $|\psi\rangle$

- Von Neumann (projective) measurement on a state is described by a set of unit-norm orthonormal vectors $\{|w_k\rangle\}$, $\langle w_k | w_j \rangle = \delta_{k,j}$

- If the state $|\psi\rangle$ is measured, the k-th outcome appears with probability, $p_k = |\langle w_k | \psi \rangle|^2$

- If two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal, i.e., $\langle \psi_1 | \psi_2 \rangle = 0$, measurement $\{ |w_1\rangle = |\psi_1\rangle, |w_2\rangle = |\psi_2\rangle \}$ achieves $P_e = 0$
Number (Fock) state of a mode

Mode $\phi(t)$, a quantum system, is excited in a coherent state $|\alpha\rangle$, $\alpha \in \mathbb{C}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons $K$ is a Poisson random variable of mean $N$.

A mode of ideal laser light is in a coherent state. Number (Fock) state of a given mode is VERY hard to produce experimentally.

There are infinitely many other types of “states” of the mode $\phi(t)$. Coherent state and Fock state are just two example class of states.

$|n\rangle$, $n \in \{0, 1, \ldots, \infty\}$ Fock states of a mode are special: they form an orthonormal basis that spans any general quantum state $|\psi\rangle$ of that mode.

$$\langle m|n\rangle = \delta_{mn} \quad \text{and} \quad |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad \sum_{n=0}^{\infty} |c_n|^2 = 1$$
Coherent state as a quantum state

\[ |\alpha\rangle = \sum_{n=0}^{\infty} \left( \frac{e^{-|\alpha|^2} \alpha^n}{\sqrt{n!}} \right) |n\rangle \]

\[ \langle \alpha|\beta \rangle = \exp \left[ \alpha^* \beta - \frac{1}{2} (|\alpha|^2 + |\beta|^2) \right] \neq 0 \]

\[ |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \ldots \]

Ideal photon detection is a von Neumann quantum measurement described by projectors, \{ |n\rangle \langle n| \}, n = 0, 1, \ldots, \infty

Ideal direct detection on a coherent state \( |\alpha\rangle \) produces outcome “n” (i.e., n “clicks”) with probability, \( p_n = |\langle n|\alpha\rangle|^2 = |c_n|^2 = \frac{e^{-N} N^n}{n!} \)

Poisson detection statistics in a laser pulse is a result of the projection of the quantum state of the laser pulse—a coherent state—on to one of the Fock states
Quantum description of light

- Complete description of an optical field is the quantum state of a set of mutually-orthogonal modes
- The most general (pure) state of a mode is an arbitrary superposition of number states of that mode, $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$
  where $\langle m | n \rangle = \delta_{mn}$, and $\sum |c_n|^2 = 1$
- Ideal direct detection: $\{|w_k\rangle\} \equiv \{|k\rangle\}$ -- number states
- Examples of pure state of a mode:
  - Coherent state $|\alpha\rangle = \sum_{n=0}^{\infty} \left( \frac{e^{-|\alpha|^2/2}}{\sqrt{n!}} \alpha^n \right) |n\rangle$, $\alpha \in \mathbb{C}$
  - Number (Fock) state, $|n\rangle$, $n \in \{1, 2, \ldots\}$
  - Cat state: $|\psi_\pm\rangle = N_\pm (|\alpha\rangle \pm | - \alpha\rangle)$
    - $N_\pm = 1/\sqrt{2(1 \pm e^{-2|\alpha|^2})}$ ensures $\langle \psi_+ | \psi_+ \rangle = 1$, and $\langle \psi_- | \psi_- \rangle = 1$ (so, the two cat states can be used to encode a qubit)
    - $\langle \psi_+ | \psi_- \rangle = 0$
Photon detection on cat states

\[ |\alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]

\[ |\alpha\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\sqrt{n!}} |n\rangle \]

\[ |\text{cat}_e\rangle \propto |\alpha\rangle + |\alpha\rangle \]

\[ |\text{cat}_e\rangle \propto 2 e^{-\frac{1}{2} |\alpha|^2} \left( \frac{\alpha^0}{\sqrt{0!}} |0\rangle + \frac{\alpha^2}{\sqrt{2!}} |2\rangle + \frac{\alpha^4}{\sqrt{4!}} |4\rangle + \ldots \right) \]

Not Poisson distribution

\[ |\text{cat}_o\rangle \propto |\alpha\rangle - |\alpha\rangle \]

\[ |\text{cat}_o\rangle \propto 2 e^{-\frac{1}{2} |\alpha|^2} \left( \frac{\alpha^1}{\sqrt{1!}} |1\rangle + \frac{\alpha^3}{\sqrt{3!}} |3\rangle + \frac{\alpha^5}{\sqrt{5!}} |5\rangle + \ldots \right) \]
Entangled states

• Product state of two modes can be written as: $|\psi_1\rangle|\psi_2\rangle$

  with, $|\psi_1\rangle = \sum_{n=0}^{\infty} a_n |n\rangle$ and $|\psi_2\rangle = \sum_{n=0}^{\infty} b_n |n\rangle$

  - Tensor product: Each state “lives in its own Hilbert space”
  - Example of product state of 2 modes:

    - Two coherent states, $|\alpha_1\rangle|\alpha_2\rangle$
    - Coherent state and a Fock state, $|\alpha\rangle|k\rangle$
    - Two cat states, $|\psi_+\rangle|\psi_-\rangle$ where $|\psi_{\pm}\rangle = N_{\pm}(|\alpha\rangle \pm |-\alpha\rangle)$

• Entangled state of two modes cannot be written as $|\psi_1\rangle|\psi_2\rangle$ A two-mode entangled state is:

  $|\psi\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c_{n_1,n_2} |n_1\rangle|n_2\rangle$, $\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |c_{n_1,n_2}|^2 = 1$

  - Example (N00N state): $|\psi\rangle = \frac{|n\rangle|0\rangle + |0\rangle|n\rangle}{\sqrt{2}}$
Binary state discrimination

\[ |\psi_1\rangle \text{ (hypothesis } H_1) \text{ vs. } |\psi_2\rangle \text{ (hypothesis } H_2) \]

- Assume equal priors: \( p_1 = p_2 = \frac{1}{2} \)

Consider a von Neumann projective measurement:

\[
\Pi_1 = |w_1\rangle\langle w_1| \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\Pi_2 = |w_2\rangle\langle w_2| 
\]

Inner product between the two states

\[ \langle \psi_1 | \psi_2 \rangle = \sigma \]

This measurement happens to be the one that minimizes the average error probability

\[
P_e = P(H_1)P(H_2|H_1) + P(H_2)P(H_1|H_2) = \frac{1}{2} \left| \langle w_2 | \psi_1 \rangle \right|^2 + \frac{1}{2} \left| \langle w_1 | \psi_2 \rangle \right|^2
\]
Minimum probability of error for binary state discrimination

\[ P_e = \frac{1}{2} |\langle w_2 | \psi_1 \rangle|^2 + \frac{1}{2} |\langle w_1 | \psi_2 \rangle|^2 \]

\[ |\psi_1 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1 + \sigma} \\ \sqrt{1 - \sigma} \end{bmatrix} \quad |w_1 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ |\psi_2 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1 + \sigma} \\ -\sqrt{1 - \sigma} \end{bmatrix} \quad |w_2 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

**Problem 3:** What is the minimum probability of error in discriminating \(|\psi_1 \rangle \) and \(|\psi_2 \rangle \) given their inner product, \(\langle \psi_1 | \psi_2 \rangle = \sigma \)? (assume \(\sigma\) is real)

A: \(P_e = \sigma\)
B: \(P_e = \frac{1 - \sqrt{1 - \sigma^2}}{2}\)
C: \(P_e = \frac{1 - \sigma}{2}\)
D: I do not know.
MPE decision among M pure states

- M-ary ensemble \( \{ p_i, |\psi_i\rangle \} ; i = 1, 2, \ldots, M \)
  - Pairwise inner products (Gram matrix), \( \sigma_{ij} = \langle \psi_i | \psi_j \rangle \)
- Yuen-Kennedy-Lax (YKL) [1975] conditions for MPE
  - Projective measurement, \( \{|\omega_j\rangle\} , j = 1, 2, \ldots, M \)
  - Relative inner products (aligning measurement vectors in the M-dimensional space spanned by the M pure states):
    \[ x_{ij} = \langle \omega_i | \psi_j \rangle \]
  - YKL conditions for minimum average probability of error

\[
\begin{align*}
\sum_{k=1}^{M} x_{k\cdot} x_{k\cdot}^* = \sigma_{ij} \\
\sum_{i=1}^{M} x_{ij} |\omega_j\rangle \langle \psi_j | \geq p_i |\psi_i\rangle \langle \psi_i |, \forall i
\end{align*}
\]

(1) \( p_m \sum_{k=1}^{M} x_{km} x_{m\cdot}^* = p_k \sum_{k=1}^{M} x_{kk} x_{m\cdot}^* \)
(2) \( \sum_{k=1}^{M} x_{kj} x_{ki}^* = \sigma_{ij} \)
(3) \( \sum_{j=1}^{M} x_{jj} |\omega_j\rangle \langle \psi_j | \geq p_i |\psi_i\rangle \langle \psi_i |, \forall i \)

\[ P_{e,\min} = 1 - \sum_{i=1}^{M} p_i |x_{ii}|^2 \]

(Check for uniqueness of solution)
BPSK – minimum probability of error

\[ |\alpha\rangle = \sum_{n=0}^{\infty} \left( \frac{e^{-|\alpha|^2/2}}{\sqrt{n!}} \alpha^n \right) |n\rangle \]

• The inner product of two coherent states \(|\alpha\rangle\) and \(|\beta\rangle\),

\[ \langle \alpha | \beta \rangle = \exp \left[ \alpha^* \beta - \frac{1}{2} (|\alpha|^2 + |\beta|^2) \right] \]

• Discriminating BPSK coherent states, \(\{|\alpha\rangle, | - \alpha\rangle\}\), \(\alpha \in \mathbb{R}\)
  
  – Inner product, \(\sigma = \langle \alpha | - \alpha \rangle = e^{-2N}, |\alpha|^2 = N\)

\[ P_{e, \text{min}} = \frac{1}{2} \left[ 1 - \sqrt{1 - |\sigma|^2} \right] = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-4N}} \right] \]

  – OOK \((|0\rangle, |\alpha\rangle), \langle 0 | \alpha \rangle = e^{-N/2}\)

\[ P_{e, \text{min}} = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-N}} \right] \]

This calculation of minimum error probability using the quantum picture was easy. How do we design a receiver that will achieve this?
Dolinar’s receiver: optimal binary detection

\[ F^2 T = N \]

Detector PPP rate:
\[ \lambda(t) = |s(t) + u_\pm(t)|^2 \]

Toggles between applying \( u_-(t) \) and \( u_+(t) \) at each detector click, and switch receiver’s belief of which hypothesis is true at each click arrival.

\[ P_e = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-4N}} \right] \]

\[ |\alpha\rangle \rightarrow |\alpha/\sqrt{K}\rangle |\alpha/\sqrt{K}\rangle |\alpha/\sqrt{K}\rangle \ldots |\alpha/\sqrt{K}\rangle \]

\[ |-\alpha\rangle \rightarrow |-\alpha/\sqrt{K}\rangle |-\alpha/\sqrt{K}\rangle |-\alpha/\sqrt{K}\rangle \ldots |-\alpha/\sqrt{K}\rangle \]
Slicing interpretation of Dolinar’s receiver
Slicing interpretation of Kennedy’s receiver (exact nulling version)
BPSK discrimination performance

Average probability of error (assuming equal priors)

\[ P_{e,\text{min}} = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-4N}} \right] \]

\( \{ |\alpha\rangle, | - \alpha\rangle \}, |\alpha|^2 = N \)

Dolinar receiver is factor of 2 (or, 3 dB) better than homodyne in the error exponent

\[ \sim e^{-4N} \text{ vs. } \sim e^{-2N} \]
Bondurant’s generalization of exact-nulling Kennedy receiver to Q-ary PSK

Q-ary PSK, Q=4 shown

Null H1
Null H1
Null H2
Null H2
Null H3

Click – rule out H1
Click – rule out H2
Optimized nulling Bondurant receiver: the equivalent of Dolinar for Q-ary PSK


They analyze performance of the receiver with all three detector imperfections we discussed.

The "quantum" limit of minimum probability of error cannot be attained by this Dolinar-like receiver strategy. A structured receiver to attain this minimum error probability remains an open problem even for 3 given states of a laser pulse!
Ternary discrimination example

- Consider $|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = |\alpha\rangle$, $|\psi_3\rangle = |-\alpha\rangle$
  - with $\alpha$ real, equal priors, $N = |\alpha|^2$; what measurement would you use to distinguish these?
- Gram matrix, $((\Gamma))_{ij} = \gamma_{ij} = \langle \psi_i | \psi_j \rangle$

$$\Gamma = \begin{pmatrix} 1 & x & x \\ x & 1 & x^4 \\ x & x^4 & 1 \end{pmatrix}, \quad x = e^{-N/2}$$

- Measurement matrix, $((X))_{ij} = x_{ij} = \langle w_i | \psi_j \rangle$

$$X = \begin{pmatrix} a & d & d \\ b & c & e \\ b & e & c \end{pmatrix}$$
Minimum probability of error

- **YKL conditions**

\[
\sum_{k=1}^{M} x_{kj} x_{ki}^* = \gamma_{ij} \quad \left\{ \begin{array}{l}
a^2 + 2b^2 = 1 \\
d^2 + c^2 + e^2 = 1 \\
ad + b(c + e) = x \\
d^2 + 2ce = x^4
\end{array} \right.
\]

\[
p_k x_{km} x_{mm}^* = p_k x_{kk} x_{mk}^* \quad \left\{ \begin{array}{l}
ab = cd
\end{array} \right.
\]

- **Solution:**

\[
a = \left[ 2xd + (1 - x^2)\sqrt{1 + x^4 - 2d^2} \right] / (1 + x^4)
\]

\[
b = \left[ x \sqrt{1 + x^4 - 2d^2} - d(1 - x^2) \right] / (1 + x^4)
\]

\[
c = \frac{1}{2} \left[ \sqrt{1 + x^4 - 2d^2} + \sqrt{1 - x^4} \right]
\]

\[
e = \frac{1}{2} \left[ \sqrt{1 + x^4 - 2d^2} - \sqrt{1 - x^4} \right]
\]

- Substitute these into \( f(d) = ab - cd \), and solve using Newton’s method for the value of \( d \), s.t. \( f(d) = 0 \). Evaluate average min probability of error, \( P_{e,\text{min}} = 1 - \frac{1}{3}(a^2 + 2c^2) \), and plot as fn. of \( N \)
Comparison of MPE with Homodyne

• Evaluate the error probability attained by an ideal homodyne detection receiver: \( X \sim \left( x_i, \frac{1}{4} \right), x_i = 0, -\alpha, \alpha \)
  – If \( |X| < -\alpha/2 \), pick \( | - \alpha \rangle \), elseif \( |X| > \alpha/2 \), pick \( |\alpha \rangle \), else pick \( |0 \rangle \)
  – Show that, \( P_{e,\text{hom}} = \frac{4}{3} \text{erfc}(\sqrt{N}) \). Let us plot this.

• Asymptotic limit (N large)
  – For N large, \( P_{e,\text{hom}} \sim e^{-N/2} \), whereas \( P_{e,\text{min}} \sim e^{-N} \)

• Kennedy like receiver
  (sequential-nulling) Kennedy receiver outperforms homodyne, and attains the optimal exponent, \( \exp(-N) \)

Plot courtesy: Quantum Detection and Estimation Theory, Carl Helstrom, 1976
Another M-ary example: pulse position modulation (PPM)

Direct detection

\[ |\psi_i\rangle = |0\rangle \ldots |0\rangle |\alpha\rangle |0\rangle \ldots |0\rangle \]

\[ |1\rangle \ldots |i\rangle \ldots |M\rangle \quad |\alpha|^2 = N \]

\[ N \text{ photon coherent-state pulses} \]

Three M=4-ary PPM frames

\[ (T/M) \quad (T_d) \quad T \]

Single photon detector response

\[ (\text{detector dead time}) \quad (\text{dark click within a pulse slot}) \]

**Problem 4:** What is the probability of error in discriminating the M-ary PPM codewords achieved by ideal direct (photon) detection on each pulse slot?

A: \[ P_e = e^{-N} \]

B: \[ P_e = e^{-2N} / M \]

C: \[ P_e = [(M - 1) / M] e^{-N} \]

D: I do not know.
Another M-ary example: pulse position modulation (PPM)

Direct detection

\[ |\psi_i\rangle = |0\rangle \ldots |0\rangle |\alpha\rangle |0\rangle \ldots |0\rangle \]
\[ |\alpha|^2 = N \]

Quantum MPE limit (YKL)

\[ \sigma_{ij} = \langle \psi_i | \psi_j \rangle = |\langle \alpha | 0 \rangle|^2 = e^{-N}, i \neq j \]
\[ x_{ij} = b, i \neq j; x_{ii} = a, \forall i \]
(symmetry postulate)

YKL conditions for minimum error probability

\[ a^2 + (M - 1)b^2 = 1 \]
\[ 2ab + (M - 2)b^2 = e^{-N} \]
\[ P_{e,\text{min}} = 1 - a^2 \]
\[ = \frac{M - 1}{M^2} \left[ \sqrt{1 + (M - 1)e^{-N}} - \sqrt{1 - e^{-N}} \right]^2 \]
\[ \sim e^{-2N}, Me^{-N} \ll 1 \]

Conditional nulling receiver

PPM demodulation using the Conditional Pulse Nulling (CPN) receiver

\[ P_{e, \text{CPN}} = \frac{1}{M} \left[ (1 - e^{-N})^M + M e^{-N} - 1 \right] \]
\[ \sim e^{-2N}, \ M e^{-N} \ll 1 \]


Receiver design that exactly attains the quantum limit is not known.
Universal quantum processing

• We know how to calculate the minimum probability of error for discriminating any M coherent states. Yet, we don’t know optimal structured receiver designs for M > 2
• The M = 2 case (Dolinar receiver) was special
• So far, we have been playing with linear optics (circuits of beamsplitters and phases) and direct (photon) detection. These are NOT universal resources
• Adding “squeezing” to our toolbox will make it universal
• *We will learn about squeezing in Module 2*
Squeezing

- Squeezing is a unitary transformation, $S(r, \theta)$

Homodyne detection results in:
- Direct detection: $P(k = 0) = \cosh(r) e^{-N(1+\tanh(r))}$

Mean photon number $|\beta|^2 + \sinh^2(r)$

Quantum result: cannot be described semi-classically

BPSK discrimination $\{|\alpha\rangle, | - \alpha\rangle\}$: generalization of Kennedy receiver (apply displacement and squeezing before photon detection)

Takeoka and Sasaki, PRA 2008
General design of an optimal receiver

• Is there a receiver strategy that uses adaptive application of squeezing (not just displacement) on small slices of the coherent state pulses, and photon detection attain arbitrary M-ary MPE state discrimination? [Open problem]

• Instead of an all-optical design, what if we can map each of the BPSK coherent states to a qubit first, i.e.,

\[ |\alpha\rangle \longrightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+\sigma} \\ \sqrt{1-\sigma} \end{bmatrix} \]

\[ |\alpha\rangle \longrightarrow |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+\sigma} \\ -\sqrt{1-\sigma} \end{bmatrix} \]

\[ \sigma = \langle -\alpha|\alpha \rangle = e^{-2N}, \quad N = |\alpha|^2 \]

– Then use quantum computing on those qubits?
Quantum limited receiver design

Optical receiver is a “mini quantum computer”

Mean photon number per pulse, $N$

Decoding error probability

Best achievable using ANY classical receiver

Quantum limit of minimum error probability

Rengaswamy, Seshadreesan, SG, Pfister, Nature npj Quantum Inf 7, 97 (2021)
Demonstration of a quantum advantage by a joint detection receiver for optical communication using quantum belief propagation on a trapped-ion device

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FIG. 5. Both decompositions for BPQM full decoder components. (a) $U$ gate decomposition, where $U_l$ is the Qiskit rotation gate and $\gamma_1$ and $\gamma_2$ are defined in Eq. (A9). (b) $K_m$ gate decomposition, utilizing ancilla qubit 3.
Communication capacity

- Mean photon number per received pulse = N
- We can excite each pulse in any coherent state, $|\alpha\rangle$, $\alpha \in \mathbb{C}$
- How many bits of information $C(N)$ can be faithfully communicated per pulse?

\[ \frac{C(N)}{N} \text{ bits per photon} \]

Bridging the remaining gap to the Holevo capacity requires joint-detection receivers that use quantum effects.


Communication capacity

- Mean photon number per received pulse = $N$
- We can excite each pulse in any coherent state, $|\alpha\rangle$, $\alpha \in \mathbb{C}$
- How many bits of information $C(N)$ can be faithfully communicated per pulse?

\[
\frac{C(N)}{N}
\]

Shannon capacity: any modulation + receiver combination
Holevo capacity: of a given modulation (optimum joint detection receiver)
Ultimate Holevo capacity: no restriction of modulation or receiver

Takeoka, SG, PRA 89, 042309 (2014)
Module 1: concluding remarks

• Laser light pulses undergo “wave-like” interference (through beamsplitters), much like ripples of water in a pond
• Laser light field cannot be precisely measured
• Detecting photons on a laser light pulse produces a random number of “clicks” with a Poisson distribution
• Quantum representation of a laser light pulse is a “coherent state” --- this representation helps us quantify “best” receivers (that minimize probability of error, for example), even without knowing how to build such as optimal receiver
• Just using semiclassical tools (interference in a beamsplitter based circuit, and Poisson-noise-limited photon detection), one cannot attain the quantum limit of receiver performance
• Optimal receiver designs require “quantum” processing of laser light – either all-optically using non-classical transformations of light (e.g., using squeezing) or first coupling the laser-light pulses into qubits, followed by processing them in a quantum computer
Break [5 minutes]
Module 2: Quantum information advantage arising from interfering photons

Outline:

1. Introduction to very basic quantum mechanics.
3. Gaussian Multiport interferometers.
4. General Gaussian Transformations and application to Gaussian Boson Sampling.
Quantum Systems

Essentially, Quantum optics is applied linear algebra.
CV System: We encode information in states of light.
Transformation of states

- Phase
- Squeezing
- Displacement

\[ \hat{U}_\beta = \exp(\beta \hat{a}^\dagger - \beta^* \hat{a}) \]
Phase space description

\[ \chi_W(\zeta^*, \zeta) = \text{tr}(\hat{\rho} e^{-\zeta^* \hat{a} + \zeta \hat{a}^\dagger}) \]
\[ \chi_A(\zeta^*, \zeta) = \text{tr}(\hat{\rho} e^{-\zeta^* \hat{a}^\dagger} e^{\zeta \hat{a}^\dagger}) \]
\[ \chi_N(\zeta^*, \zeta) = \text{tr}(\hat{\rho} e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}) \]

**Characteristic functions**

\[ \chi_A(\zeta) = \int Q(\alpha) e^{\zeta \alpha^* - \zeta^* \alpha} d^2 \alpha \]
\[ Q(\alpha) = \frac{1}{\pi^2} \int \chi_A(\zeta) e^{-\zeta \alpha^* + \zeta^* \alpha} d^2 \zeta \]

\[ \chi_W(\zeta) = \int W(\alpha) e^{\zeta \alpha^* - \zeta^* \alpha} d^2 \alpha \]
\[ W(\alpha) = \frac{1}{\pi^2} \int \chi_W(\zeta) e^{-\zeta \alpha^* + \zeta^* \alpha} d^2 \zeta \]

**Q-Function:** Always a proper probability density function

May not be a proper probability density function. **Can take negative values.**

Always a proper probability density function **when it exists.** The states for which a proper P function exists are classical states.
Unitary evolution

\[
\begin{align*}
\chi_W(\zeta^*, \zeta) &= \text{tr}(\rho e^{-\zeta^* \hat{a}} + \zeta \hat{a}^\dagger) \\
\chi_A(\zeta^*, \zeta) &= \text{tr}(\rho e^{-\zeta^* \hat{a}} e^{\zeta \hat{a}^\dagger}) \\
\chi_N(\zeta^*, \zeta) &= \text{tr}(\rho e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}})
\end{align*}
\]

**Gaussian Unitary Operator:** Any unitary operator that maps Gaussian states to Gaussian states.
Two-mode transformation: Beam splitter

Beam splitter

\[ \hat{U}_{BS} = e^{i\theta (\hat{a} \hat{b}^\dagger + \hat{a}^\dagger \hat{b})} \]

\[ \hat{\rho}_{\text{in}} \quad \hat{\rho}_{\text{out}} \]

\[ \hat{a}_{\text{out}} = U^\dagger \hat{a}_{\text{in}} \ U = \sqrt{\tau} \hat{a}_{\text{in}} + \sqrt{1 - \tau} \hat{b}_{\text{in}} \]

\[ \hat{b}_{\text{out}} = U^\dagger \hat{b}_{\text{in}} \ U = -\sqrt{1 - \tau} \hat{a}_{\text{in}} + \sqrt{\tau} \hat{b}_{\text{in}} \]

Two-mode squeezed vacuum (TMSV)
Examples of States of Light

- Classical and **Gaussian**
- Classical and **Non-Gaussian**
- Quantum and **Gaussian**
- Quantum and **Non-Gaussian**

Mixture of coherent states under a non-Gaussian PDF:
Gaussian transformations not universal.

Need any one non-Gaussian unitary

- Phase (1 \to 1)
- Beam splitter (2 \to 2)
  \[ \hat{U}_{BS} = e^{i\theta (\hat{a}^{\dagger} + \hat{b}^{\dagger})} \]
- Squeezing (1 \to 1)
- Displacement (1 \to 1)
  \[ \hat{U}_\beta = \exp(\beta \hat{a}^{\dagger} - \beta^* \hat{a}) \]
- Cubic phase (1 \to 1)
  \[ U(\gamma) = e^{i\gamma \hat{q}^3}, \hat{q} = \frac{\hat{a} + \hat{a}^{\dagger}}{\sqrt{2}} \]


Purely Quantum properties

1. Squeezing (Gaussian Quantum resources).
2. Entanglement.
3. Non-Gaussian states and measurements.

- Sensing
- Communications.
- Universal Quantum Computation
Example on beam splitter calculations

\[ \hat{a}_{\text{out}} = U^\dagger \hat{a}_{\text{in}} U = \sqrt{\tau} \hat{a}_{\text{in}} + \sqrt{1 - \tau} \hat{b}_{\text{in}} \]

\[ \hat{b}_{\text{out}} = U^\dagger \hat{b}_{\text{in}} U = -\sqrt{1 - \tau} \hat{a}_{\text{in}} + \sqrt{\tau} \hat{b}_{\text{in}} \]
A coherent state input in a beam splitter \textbf{cannot} produce entanglement: The resulting state is a product of two coherent states (while the total mean photon number is conserved).
More examples on beam splitter calculations

\[ \hat{a}_{\text{out}} = U^\dagger \hat{a}_{\text{in}} U = \sqrt{\tau} \hat{a}_{\text{in}} + \sqrt{1 - \tau} \hat{b}_{\text{in}} \]

\[ \hat{b}_{\text{out}} = U^\dagger \hat{b}_{\text{in}} U = -\sqrt{1 - \tau} \hat{a}_{\text{in}} + \sqrt{\tau} \hat{b}_{\text{in}} \]
Problem 1: $|1\rangle \rightarrow \text{?} |\Phi_{11}\rangle$

$|1\rangle \rightarrow \tau = \frac{1}{3}$

A: $|\Phi_{11}\rangle = -\frac{2}{3} |20\rangle - \frac{1}{3} |11\rangle + \frac{2}{3} |02\rangle$

B: $|\Phi_{11}\rangle = -\frac{1}{2} |20\rangle + \frac{1}{2} |02\rangle$

C: $|\Phi_{11}\rangle = \frac{1}{2} |20\rangle + \frac{1}{2} |02\rangle$

D: I do not know
Problem 2:

\[ |1\rangle \quad \xrightarrow{\tau = \frac{1}{2}} \quad |\psi_{100}\rangle \]

\[ |0\rangle \quad \xrightarrow{\tau = \frac{1}{2}} \quad |\psi_{100}\rangle \]

A: \[ |\psi_{100}\rangle = \frac{1}{\sqrt{2}} |100\rangle + \frac{1}{2} |010\rangle + \frac{1}{2} |001\rangle \]

B: \[ |\psi_{100}\rangle = \frac{1}{\sqrt{2}} |110\rangle \]

C: \[ |\psi_{100}\rangle = \frac{1}{2} |200\rangle + \frac{1}{2} |002\rangle \]

D: I do not know.
Break [5 minutes]
Introducing multi-mode Gaussian interferometers

We have seen that the beam-splitter transforms the annihilation and creation as:

\[
\begin{align*}
\hat{a}_{\text{out}} &= U^\dagger \hat{a}_{\text{in}} U = \sqrt{\tau} \hat{a}_{\text{in}} + \sqrt{1 - \tau} \hat{b}_{\text{in}} \\
\hat{b}_{\text{out}} &= U^\dagger \hat{b}_{\text{in}} U = -\sqrt{1 - \tau} \hat{a}_{\text{in}} + \sqrt{\tau} \hat{b}_{\text{in}}
\end{align*}
\]

\[
\begin{align*}
\hat{a}^\dagger_{\text{out}} &= \sqrt{\tau} \hat{a}^\dagger_{\text{in}} + \sqrt{1 - \tau} \hat{b}^\dagger_{\text{in}} \\
\hat{b}^\dagger_{\text{out}} &= -\sqrt{1 - \tau} \hat{a}^\dagger_{\text{in}} + \sqrt{\tau} \hat{b}^\dagger_{\text{in}}
\end{align*}
\]

\[
\begin{align*}
\hat{b}_1 &= \sqrt{\tau} \hat{a}_1 + \sqrt{1 - \tau} \hat{a}_2 \\
\hat{b}_2 &= -\sqrt{1 - \tau} \hat{a}_1 + \sqrt{\tau} \hat{a}_2
\end{align*}
\]

\[
\begin{align*}
\hat{b}^\dagger_1 &= \sqrt{\tau} \hat{a}^\dagger_1 + \sqrt{1 - \tau} \hat{a}^\dagger_2 \\
\hat{b}^\dagger_2 &= -\sqrt{1 - \tau} \hat{a}^\dagger_1 + \sqrt{\tau} \hat{a}^\dagger_2
\end{align*}
\]
Multimode Gaussian Unitary evolution

For example: The beam splitter corresponds to:

\[
\begin{pmatrix}
\hat{a} \\
\hat{b} \\
\hat{a}^\dagger \\
\hat{b}^\dagger
\end{pmatrix} = V
\begin{pmatrix}
\hat{a} \\
\hat{b} \\
\hat{a}^\dagger \\
\hat{b}^\dagger
\end{pmatrix}
\]

\[
V = \begin{pmatrix}
\sqrt{\tau} & \sqrt{1-\tau} & 0 & 0 \\
-\sqrt{1-\tau} & \sqrt{\tau} & 0 & 0 \\
0 & 0 & \sqrt{\tau} & \sqrt{1-\tau} \\
0 & 0 & -\sqrt{1-\tau} & \sqrt{\tau}
\end{pmatrix}
\]
Multimode Gaussian Unitary evolution

\[
\begin{pmatrix}
\hat{b} \\
\hat{b}^\dagger
\end{pmatrix} = V \begin{pmatrix}
\hat{a} \\
\hat{a}^\dagger
\end{pmatrix}, \quad \begin{pmatrix}
\hat{b}^\dagger \\
\hat{b} \\
\hat{b}^\dagger
\end{pmatrix} = \begin{pmatrix}
\hat{a}^\dagger \\
\hat{a}
\end{pmatrix} V^\dagger V \begin{pmatrix}
\hat{a} \\
\hat{a}^\dagger
\end{pmatrix} \Rightarrow 2\hat{b}^\dagger \hat{b} + 1 = \begin{pmatrix}
\hat{a}^\dagger \\
\hat{a}
\end{pmatrix} V^\dagger V \begin{pmatrix}
\hat{a} \\
\hat{a}^\dagger
\end{pmatrix}
\]
Problem 3: Let the single mode squeezer be described by $V = \begin{pmatrix} \cosh r & -\sinh r \\ -\sinh r & \cosh r \end{pmatrix}$

Is $V$ unitary?

A: No.
B: Yes.
C: Depends on the state we transform.
D: I do not know.
Problem 4: Let the single mode squeezed vacuum state

\[ \hat{G}_{\text{squeezer}}|0\rangle = |0; r\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \left( -\frac{1}{2} \tan hr \right)^n |2n\rangle \]

What is the mean photon number of said state?
Reminder: Mean photon number \( \bar{n} = \langle 0; r | \hat{n} | 0; r \rangle \) and \( \hat{n} |n\rangle = n |n\rangle \)

A: \( \bar{n} = 0. \)
B: \( \bar{n} = \sinh^2 r. \)
C: \( \bar{n} = \frac{1}{2} \sinh^2 r. \)
D: I do not know.
Squeezed states: A quantum resource

With the math you’ve learned you can calculate:

Two-mode squeezed vacuum (TMSV)

The TMSV is the maximally entangled two-mode Gaussian state.

A more general transformation:
The probability of any PNR pattern

\[ P_{\vec{n}} = \frac{|I_{n_1 \ldots n_N}|^2}{\prod_{i=1}^{N} n_i! 2^{n_i} \cosh r_i} \]

\[ I_{\vec{n}} = \int d^N q_\alpha d^N p_\alpha R(\vec{x}_\alpha) \prod_{i=1}^{N} (q_{\alpha_i} + i p_{\alpha_i})^{n_i} = \langle f_1^{n_1} \ldots f_N^{n_N} \rangle, \quad f_i = q_{\alpha_i} + i p_{\alpha_i} \]

\[ R(\vec{x}_\alpha) = \frac{1}{(2\pi)^N} e^{-\frac{1}{2} \vec{x}_\alpha^T \mathcal{H} \vec{x}_\alpha} \]
The probability of any PNR pattern: Hafnians

The difficult part in calculating the probability of a photon-number pattern is the term:

\[
\mathcal{I}_n = \int d^N q \, d^N p \, R(\vec{x}_\alpha) \prod_{i=1}^{N} (q_{\alpha_i} + i p_{\alpha_i})^{n_i} = \langle f_1^{n_1} \ldots f_N^{n_N} \rangle, \quad f_i = q_{\alpha_i} + i p_{\alpha_i}
\]

Hafnian of matrix F:

\[
\mathcal{H}_n = \begin{cases} 
0 & \Sigma = \text{odd}, \\
\text{Hf}(F) & \Sigma = \text{even}
\end{cases} \quad \Sigma = \sum_{i=1}^{N} n_i
\]
In general, a (classical) computer program that samples from a given distribution would scale exponentially with the size of the requested pattern (e.g. photon-number pattern). Therefore, we program the Hafnian into a quantum-optical circuit, i.e., a *quantum Galton board* (a.k.a. Gaussian Boson Sampler) that performs the sampling job.
Hafnians and perfect matchings

Set of objects

**Perfect matching:** Connect the objects with a line so that any object is used only once (no more than one line can start/end from any point).

Example:

Perfect matching 1  Perfect matching 2  Perfect matching 3
Hafnians and perfect matchings

How does the previous example relate to the calculation of the Hafnian?
Isserlis’ theorem

\[
P_\vec{n} = \frac{|\mathcal{I}_{n_1 \ldots n_N}|^2}{\prod_{i=1}^{N} n_i!2^{n_i} \cosh r_i}
\]

\[
\mathcal{I}_n = \int d^N q_\alpha d^N p_\alpha R(\vec{x}_\alpha) \prod_{i=1}^{N} (q_{\alpha_i} + i p_{\alpha_i})^{n_i} = \langle f_1^{n_1} \ldots f_N^{n_N} \rangle, f_i = q_{\alpha_i} + i p_{\alpha_i}
\]

\[
\langle g_1 g_2 \cdots g_{\Sigma} \rangle = \sum_{p \in P_{\Sigma}^2} \prod_{\{i,j\} \in p} \langle g_i g_j \rangle
\]
Module 2: Concluding remarks

- Quantum states of light can:
  - Possess counter-intuitive properties
  - Give quantum advantage

- Necessary tools to manipulate all the different protocols:
  - Linear algebra.
  - Building intuition.
Quantum supremacy (advantage)

• Google, US
  - IQP: a random circuit based sampler [53-qubit circuit of depth 20, with 430 two-qubit and 1,113 single-qubit gates. Classical simulation estimate ~ 10,000 years based on a Schrödinger-Feynman simulation that trades off space for time, whereas Sycamore processor takes about 200 seconds

• Xanadu, Canada
  - Gaussian Boson Sampling: 216 squeezed modes entangled with three-dimensional connectivity; it would take ~ 9,000 years for the best available algorithms and supercomputers to produce, using exact methods, a single sample from the programmed distribution, whereas Borealis requires only 36 μs

• USTC, China
  - Gaussian Boson Sampling: 50 squeezed modes in a 100-mode interferometer; measured a sampling rate that is about $10^{14}$-fold faster than using state-of-the-art classical simulation strategies and supercomputers (2.5 billion years as opposed to 200 s)
What are random samplers good for?

- Quantum generative adversarial networks (arXiv:1804.08641)
- Graph isomorphism and graph similarity testing (arXiv:1810.10644)
- Molecular vibronic spectra for quantum chemistry calculations (arXiv:1912.07634)
- Quantum adiabatic optimization algorithm or QAOA (arXiv:1902.00409)
- Producing samples from hard-to-generate stochastic point processes (arXiv:1906.11972)

Random samplers (Boson sampling, IQP, GBS, etc.) are not universal quantum computers. But they are believed to be strictly more powerful than classical computers.
Conclusion

• The building blocks we covered in Modules 1 and 2 (linear optics, coherent states, multi-photon states, squeezed states, photon detection, homodyne detection) in principle suffice to design “quantum-optimal” transmitters, processors, computers, receivers, for all applications in photonics-based information processing.

• Yet, we don’t know how to “put together” these building blocks to attain best performance in most applications of photonic information processing!
Course Evaluation Survey

We value your feedback on all aspects of this short course. Please go to the link provided in the Zoom Chat or in the email you will soon receive to give your opinions of what worked and what could be improved.

CQN Winter School on Quantum Networks

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